Short definitions in constraint languages

Jakub Bulín[a], joint work with Michael Kompatscher

105. Arbeitstagung Allgemeine Algebra
Prague, June 2, 2024

[a] Supported by Charles University project UNCE/SCI/004 & MEYS Inter-excellence project LTAUSA19070

[1] J. Bulín and M. Kompatscher: *Short definitions in constraint languages*, 48th International Symposium on Mathematical Foundations of Computer Science (MFCS 2023)

The motivation

Some properties of linear algebra (over a finite field)

Some properties of linear algebra (over a finite field)

- few subspaces

Some properties of linear algebra (over a finite field)

- few subspaces, $2^{O(n^2)}$

Some properties of linear algebra (over a finite field)

- few subspaces, $2^{O(n^2)}$
- small generating sets

Some properties of linear algebra (over a finite field)

- few subspaces, $2^{O(n^2)}$
- small generating sets, $O(n^2)$

Some properties of linear algebra (over a finite field)

- few subspaces, $2^{O(n^2)}$
- small generating sets, $O(n^2)$
- subspace membership in P

Some properties of linear algebra (over a finite field)

- few subspaces, $2^{O(n^2)}$
- small generating sets, $O(n^2)$
- subspace membership in P (echelon form + row-reduce)

Some properties of linear algebra (over a finite field)

- few subspaces, $2^{O(n^2)}$
- small generating sets, $O(n^2)$
- subspace membership in P (echelon form + row-reduce)

- short definitions, $O(n^2)$

Some properties of linear algebra (over a finite field)

- few subspaces, $2^{O(n^2)}$
- small generating sets, $O(n^2)$
- subspace membership in P (echelon form + row-reduce)

- short definitions, $O(n^2)$ ... express any linear system using $x + y = z, x = 0, x = 1$, $\exists$ (auxiliary vars), $\wedge$ (conjunction)

Some properties of finite abelian groups

Some properties of finite abelian groups

- few sub[groups of]powers

Some properties of finite abelian groups

- few sub[groups of]powers
- small generating sets

Some properties of finite abelian groups

- few sub[groups of]powers
- small generating sets
- subpower membership in P

Some properties of finite abelian groups

- few sub[groups of]powers
- small generating sets
- subpower membership in P

- short definitions

Some properties of finite groups

Some properties of finite groups

- few subpowers

Some properties of finite groups

- few subpowers
- small generating sets

Some properties of finite groups

- few subpowers
- small generating sets
- subpower membership in P

Some properties of finite groups

- few subpowers
- small generating sets
- subpower membership in P (Schreier-Sims: SGS + sifting)

Some properties of finite groups

- few subpowers
- small generating sets
- subpower membership in P (Schreier-Sims: SGS + sifting)

- short definitions??

Some properties of finite Mal'tsev algebras

Some properties of finite Mal'tsev algebras

- few subpowers

Some properties of finite Mal'tsev algebras

- few subpowers
- small generating sets

Some properties of finite Mal'tsev algebras

- few subpowers
- small generating sets
- subpower membership in NP

Some properties of finite Mal'tsev algebras

- few subpowers
- small generating sets
- subpower membership in NP, is it in P??

Some properties of finite Mal'tsev algebras

- few subpowers
- small generating sets
- subpower membership in NP, is it in P??

- short definitions??

The what and the why

Explaining the title

Explaining the title

“. . . constraint languages”

Explaining the title

"... constraint languages"

- A constraint language over a finite domain A:

$$\Gamma = \{R_1, \ldots, R_m\} \text{ where } R_i \subseteq A^{n_i}$$

Explaining the title

"... constraint languages"

- A constraint language over a finite domain A:

$$\Gamma = \{R_1, \ldots, R_m\} \text{ where } R_i \subseteq A^{n_i}$$

- **Example (2-SAT)** $A = \{0, 1\}$, $\Gamma_{\mathrm{2SAT}} = \{R_{00}, R_{01}, R_{10}, R_{11}\}$ where $R_{ij} = \{0, 1\}^2 \setminus \{(i, j)\}$ (e.g. R_{01} encodes $x \vee \neg y$)

Explaining the title

"...constraint languages"

- A constraint language over a finite domain A:

$$\Gamma = \{R_1, \dots, R_m\} \text{ where } R_i \subseteq A^{n_i}$$

- **Example (2-SAT)** $A = \{0, 1\}$, $\Gamma_{\mathrm{2SAT}} = \{R_{00}, R_{01}, R_{10}, R_{11}\}$ where $R_{ij} = \{0, 1\}^2 \setminus \{(i, j)\}$ (e.g. R_{01} encodes $x \vee \neg y$)

"...definitions in..."

Explaining the title

"...constraint languages"

- A constraint language over a finite domain A:

$$\Gamma = \{R_1, \ldots, R_m\} \text{ where } R_i \subseteq A^{n_i}$$

- **Example (2-SAT)** $A = \{0, 1\}$, $\Gamma_{\mathrm{2SAT}} = \{R_{00}, R_{01}, R_{10}, R_{11}\}$ where $R_{ij} = \{0, 1\}^2 \setminus \{(i, j)\}$ (e.g. R_{01} encodes $x \vee \neg y$)

"...definitions in..."

- A primitive positive (pp-) formula: $\exists, \wedge, =$ and symbols from Γ

Explaining the title

"...constraint languages"

- A constraint language over a finite domain A:

$$\Gamma = \{R_1, \ldots, R_m\} \text{ where } R_i \subseteq A^{n_i}$$

- **Example (2-SAT)** $A = \{0, 1\}$, $\Gamma_{\mathrm{2SAT}} = \{R_{00}, R_{01}, R_{10}, R_{11}\}$ where $R_{ij} = \{0, 1\}^2 \setminus \{(i, j)\}$ (e.g. R_{01} encodes $x \vee \neg y$)

"...definitions in..."

- A primitive positive (pp-) formula: $\exists, \wedge, =$ and symbols from Γ
- A pp-definition: $\phi(x_1, \ldots, x_n)$ defines $R \subseteq A^n$ in the usual way

Explaining the title

"...constraint languages"

- A constraint language over a finite domain A:

$$\Gamma = \{R_1, \dots, R_m\} \text{ where } R_i \subseteq A^{n_i}$$

- **Example (2-SAT)** $A = \{0,1\}$, $\Gamma_{\mathrm{2SAT}} = \{R_{00}, R_{01}, R_{10}, R_{11}\}$ where $R_{ij} = \{0,1\}^2 \setminus \{(i,j)\}$ (e.g. R_{01} encodes $x \vee \neg y$)

"...definitions in..."

- A primitive positive (pp-) formula: $\exists, \wedge, =$ and symbols from Γ
- A pp-definition: $\phi(x_1, \dots, x_n)$ defines $R \subseteq A^n$ in the usual way
- The relational clone: $\langle \Gamma \rangle = \{R \mid R \text{ is pp-definable from } \Gamma\}$

Explaining the title

"...constraint languages"

- A constraint language over a finite domain A:

$$\Gamma = \{R_1, \ldots, R_m\} \text{ where } R_i \subseteq A^{n_i}$$

- **Example (2-SAT)** $A = \{0,1\}$, $\Gamma_{\mathrm{2SAT}} = \{R_{00}, R_{01}, R_{10}, R_{11}\}$ where $R_{ij} = \{0,1\}^2 \setminus \{(i,j)\}$ (e.g. R_{01} encodes $x \vee \neg y$)

"...definitions in..."

- A primitive positive (pp-) formula: $\exists, \wedge, =$ and symbols from Γ
- A pp-definition: $\phi(x_1, \ldots, x_n)$ defines $R \subseteq A^n$ in the usual way
- The relational clone: $\langle\Gamma\rangle = \{R \mid R \text{ is pp-definable from } \Gamma\}$

"Short..."

Explaining the title

"…constraint languages"

- A constraint language over a finite domain A:

$$\Gamma = \{R_1, \dots, R_m\} \text{ where } R_i \subseteq A^{n_i}$$

- **Example (2-SAT)** $A = \{0,1\}$, $\Gamma_{\mathrm{2SAT}} = \{R_{00}, R_{01}, R_{10}, R_{11}\}$ where $R_{ij} = \{0,1\}^2 \setminus \{(i,j)\}$ (e.g. R_{01} encodes $x \vee \neg y$)

"…definitions in…"

- A primitive positive (pp-) formula: $\exists, \wedge, =$ and symbols from Γ
- A pp-definition: $\phi(x_1, \dots, x_n)$ defines $R \subseteq A^n$ in the usual way
- The relational clone: $\langle\Gamma\rangle = \{R \mid R \text{ is pp-definable from } \Gamma\}$

"Short…"

- Each $R \in \langle\Gamma\rangle_n$ has a pp-definition of length polynomial in n

Motivation: constraint satisfaction

Motivation: constraint satisfaction

$\mathrm{CSP}(\Gamma)$

input a pp-sentence Φ over Γ

question $\Gamma \models \Phi$?

Motivation: constraint satisfaction

$\mathrm{CSP}(\Gamma)$

input a pp-sentence Φ over Γ

question $\Gamma \models \Phi$?

Example The 2-CNF formula $(x \vee \neg y) \wedge (y \vee z) \wedge (z \vee \neg x)$ is encoded as $\Phi = (\exists x)(\exists y)(\exists z)(R_{01}(x, y) \wedge R_{00}(y, z) \wedge R_{01}(z, x))$

Motivation: constraint satisfaction

$\mathrm{CSP}(\Gamma)$

input a pp-sentence Φ over Γ

question $\Gamma \models \Phi$?

Example The 2-CNF formula $(x \vee \neg y) \wedge (y \vee z) \wedge (z \vee \neg x)$ is encoded as $\Phi = (\exists x)(\exists y)(\exists z)(R_{01}(x,y) \wedge R_{00}(y,z) \wedge R_{01}(z,x))$

Moreover:

Motivation: constraint satisfaction

$\mathrm{CSP}(\Gamma)$

input a pp-sentence Φ over Γ

question $\Gamma \models \Phi$?

Example The 2-CNF formula $(x \vee \neg y) \wedge (y \vee z) \wedge (z \vee \neg x)$ is encoded as $\Phi = (\exists x)(\exists y)(\exists z)(R_{01}(x, y) \wedge R_{00}(y, z) \wedge R_{01}(z, x))$

Moreover:

- solution sets are pp-definable

Motivation: constraint satisfaction

$\mathrm{CSP}(\Gamma)$

input a pp-sentence Φ over Γ

question $\Gamma \models \Phi$?

Example The 2-CNF formula $(x \vee \neg y) \wedge (y \vee z) \wedge (z \vee \neg x)$ is encoded as $\Phi = (\exists x)(\exists y)(\exists z)(R_{01}(x, y) \wedge R_{00}(y, z) \wedge R_{01}(z, x))$

Moreover:

- solution sets are pp-definable
- pp-definitions are gadget reductions

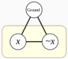
Motivation: constraint satisfaction

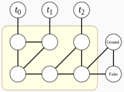
CSP(Γ)

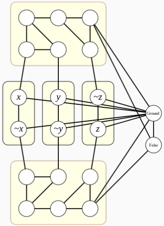
input a pp-sentence Φ over Γ

question $\Gamma \models \Phi$?

Example The 2-CNF formula $(x \vee \neg y) \wedge (y \vee z) \wedge (z \vee \neg x)$ is encoded as $\Phi = (\exists x)(\exists y)(\exists z)(R_{01}(x, y) \wedge R_{00}(y, z) \wedge R_{01}(z, x))$







Moreover:

- solution sets are pp-definable
- pp-definitions are gadget reductions

$\mathrm{CSP}(\Gamma)$

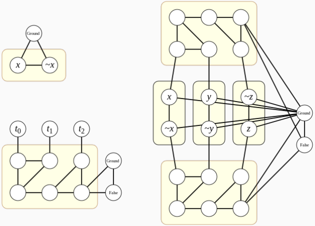
input a pp-sentence Φ over Γ

question $\Gamma \models \Phi$?

Example The 2-CNF formula $(x \vee \neg y) \wedge (y \vee z) \wedge (z \vee \neg x)$ is encoded as $\Phi = (\exists x)(\exists y)(\exists z)(R_{01}(x, y) \wedge R_{00}(y, z) \wedge R_{01}(z, x))$

Moreover:

- solution sets are pp-definable
- pp-definitions are gadget reductions



Theorem (Jeavons, Cohen, Gyssens JACM 1997)
If $\Delta \subseteq \langle \Gamma \rangle$, then $\mathrm{CSP}(\Delta)$ reduces to $\mathrm{CSP}(\Gamma)$.

The examples

Nonexamples and boring examples

Nonexamples and boring examples

Γ has short definitions, if $\exists$ polynomial $p(n)$ such that each $R \in \langle\Gamma\rangle_n$ has a pp-definition $\phi(x_1, \ldots, x_n)$ of length $|\phi| \le p(n)$.

Nonexamples and boring examples

Γ has short definitions, if $\exists$ polynomial $p(n)$ such that each $R \in \langle\Gamma\rangle_n$ has a pp-definition $\phi(x_1, \ldots, x_n)$ of length $|\phi| \leq p(n)$.

Nonexamples (3-SAT, Horn-SAT)

Nonexamples and boring examples

Γ has short definitions, if $\exists$ polynomial $p(n)$ such that each $R \in \langle \Gamma \rangle_n$ has a pp-definition $\phi(x_1, \ldots, x_n)$ of length $|\phi| \leq p(n)$.

Nonexamples (3-SAT, Horn-SAT)

Cardinality argument: short definitions $\Rightarrow \langle \Gamma \rangle_n \in 2^{O(n^k)}$

Nonexamples and boring examples

Γ has short definitions, if $\exists$ polynomial $p(n)$ such that each $R \in \langle\Gamma\rangle_n$ has a pp-definition $\phi(x_1, \dots, x_n)$ of length $|\phi| \leq p(n)$.

Nonexamples (3-SAT, Horn-SAT)

Cardinality argument: short definitions $\Rightarrow \langle\Gamma\rangle_n \in 2^{O(n^k)}$

- Γ_{3SAT} doesn't have short definitions, $\langle\Gamma_{\mathrm{3SAT}}\rangle_n$ contains all 2^{2^n} n-ary relations

Nonexamples and boring examples

Γ has short definitions, if $\exists$ polynomial $p(n)$ such that each $R \in \langle\Gamma\rangle_n$ has a pp-definition $\phi(x_1, \ldots, x_n)$ of length $|\phi| \leq p(n)$.

Nonexamples (3-SAT, Horn-SAT)

Cardinality argument: short definitions $\Rightarrow \langle\Gamma\rangle_n \in 2^{O(n^k)}$

- Γ_{3SAT} doesn't have short definitions, $\langle\Gamma_{\mathrm{3SAT}}\rangle_n$ contains all 2^{2^n} n-ary relations
- Similarly for $\Gamma_{\mathrm{HornSAT}}$, $|\langle\Gamma_{\mathrm{HornSAT}}\rangle|_n$ is double exponential

Nonexamples and boring examples

Γ has short definitions, if $\exists$ polynomial $p(n)$ such that each $R \in \langle \Gamma \rangle_n$ has a pp-definition $\phi(x_1, \ldots, x_n)$ of length $|\phi| \leq p(n)$.

Nonexamples (3-SAT, Horn-SAT)

Cardinality argument: short definitions $\Rightarrow \langle \Gamma \rangle_n \in 2^{O(n^k)}$

- Γ_{3SAT} doesn't have short definitions, $\langle \Gamma_{\mathrm{3SAT}} \rangle_n$ contains all 2^{2^n} n-ary relations
- Similarly for $\Gamma_{\mathrm{HornSAT}}$, $|\langle \Gamma_{\mathrm{HornSAT}} \rangle|_n$ is double exponential

Boring example (2-SAT)

Nonexamples and boring examples

Γ has short definitions, if $\exists$ polynomial $p(n)$ such that each $R \in \langle\Gamma\rangle_n$ has a pp-definition $\phi(x_1, \ldots, x_n)$ of length $|\phi| \le p(n)$.

Nonexamples (3-SAT, Horn-SAT)

Cardinality argument: short definitions $\Rightarrow \langle\Gamma\rangle_n \in 2^{O(n^k)}$

- Γ_{3SAT} doesn't have short definitions, $\langle\Gamma_{\mathrm{3SAT}}\rangle_n$ contains all 2^{2^n} n-ary relations
- Similarly for $\Gamma_{\mathrm{HornSAT}}$, $|\langle\Gamma_{\mathrm{HornSAT}}\rangle|_n$ is double exponential

Boring example (2-SAT)

Γ_{2SAT} has short definitions:

Nonexamples and boring examples

Γ has short definitions, if $\exists$ polynomial $p(n)$ such that each $R \in \langle \Gamma \rangle_n$ has a pp-definition $\phi(x_1, \ldots, x_n)$ of length $|\phi| \leq p(n)$.

Nonexamples (3-SAT, Horn-SAT)

Cardinality argument: short definitions $\Rightarrow \langle \Gamma \rangle_n \in 2^{O(n^k)}$

- Γ_{3SAT} doesn't have short definitions, $\langle \Gamma_{\mathrm{3SAT}} \rangle_n$ contains all 2^{2^n} n-ary relations
- Similarly for $\Gamma_{\mathrm{HornSAT}}$, $|\langle \Gamma_{\mathrm{HornSAT}} \rangle|_n$ is double exponential

Boring example (2-SAT)

Γ_{2SAT} has short definitions: each $R \in \langle \Gamma_{\mathrm{2SAT}} \rangle_n$ satisfies the 2-Helly property

Nonexamples and boring examples

Γ has short definitions, if $\exists$ polynomial $p(n)$ such that each $R \in \langle \Gamma \rangle_n$ has a pp-definition $\phi(x_1, \ldots, x_n)$ of length $|\phi| \le p(n)$.

Nonexamples (3-SAT, Horn-SAT)

Cardinality argument: short definitions $\Rightarrow \langle \Gamma \rangle_n \in 2^{O(n^k)}$

- Γ_{3SAT} doesn't have short definitions, $\langle \Gamma_{\mathrm{3SAT}} \rangle_n$ contains all 2^{2^n} n-ary relations
- Similarly for $\Gamma_{\mathrm{HornSAT}}$, $|\langle \Gamma_{\mathrm{HornSAT}} \rangle|_n$ is double exponential

Boring example (2-SAT)

Γ_{2SAT} has short definitions: each $R \in \langle \Gamma_{\mathrm{2SAT}} \rangle_n$ satisfies the 2-Helly property (and binary relations are pp-definable from Γ_{2SAT}):

Nonexamples and boring examples

Γ has short definitions, if $\exists$ polynomial $p(n)$ such that each $R \in \langle\Gamma\rangle_n$ has a pp-definition $\phi(x_1, \ldots, x_n)$ of length $|\phi| \leq p(n)$.

Nonexamples (3-SAT, Horn-SAT)

Cardinality argument: short definitions $\Rightarrow \langle\Gamma\rangle_n \in 2^{O(n^k)}$

- Γ_{3SAT} doesn't have short definitions, $\langle\Gamma_{\mathrm{3SAT}}\rangle_n$ contains all 2^{2^n} n-ary relations
- Similarly for $\Gamma_{\mathrm{HornSAT}}$, $|\langle\Gamma_{\mathrm{HornSAT}}\rangle|_n$ is double exponential

Boring example (2-SAT)

Γ_{2SAT} has short definitions: each $R \in \langle\Gamma_{\mathrm{2SAT}}\rangle_n$ satisfies the 2-Helly property (and binary relations are pp-definable from Γ_{2SAT}):

$$R(x_1, \ldots, x_n) \;\leftrightarrow\; \bigwedge_{1 \leq i \leq j \leq n} \mathrm{pr}_{ij} R(x_i, x_j)$$

Nonexamples and boring examples

Γ has short definitions, if $\exists$ polynomial $p(n)$ such that each $R \in \langle\Gamma\rangle_n$ has a pp-definition $\phi(x_1, \ldots, x_n)$ of length $|\phi| \leq p(n)$.

Nonexamples (3-SAT, Horn-SAT)

Cardinality argument: short definitions $\Rightarrow \langle\Gamma\rangle_n \in 2^{O(n^k)}$

- Γ_{3SAT} doesn't have short definitions, $\langle\Gamma_{\mathrm{3SAT}}\rangle_n$ contains all 2^{2^n} n-ary relations
- Similarly for $\Gamma_{\mathrm{HornSAT}}$, $|\langle\Gamma_{\mathrm{HornSAT}}\rangle|_n$ is double exponential

Boring example (2-SAT)

Γ_{2SAT} has short definitions: each $R \in \langle\Gamma_{\mathrm{2SAT}}\rangle_n$ satisfies the 2-Helly property (and binary relations are pp-definable from Γ_{2SAT}):

$$R(x_1, \ldots, x_n) \;\leftrightarrow \bigwedge_{1 \leq i \leq j \leq n} \mathrm{pr}_{ij} R(x_i, x_j)$$

$\Rightarrow$ pp-definitions of length $O(n^2)$

Interesting example

Interesting example (Linear systems over $\mathbb{Z}_2$)

Interesting example

Interesting example (Linear systems over $\mathbb{Z}_2$)

- $A = \{0, 1\}$, $\Gamma_{\mathrm{Lin}} = \{R_{\mathrm{Lin}}, C_0, C_1\}$ where $C_a = \{a\}$ and

$$R_{\mathrm{Lin}} = \{(a, b, c) \in \{0, 1\}^3 \mid a + b = c\}$$

Interesting example

Interesting example (Linear systems over $\mathbb{Z}_2$)

- $A = \{0, 1\}$, $\Gamma_{\mathrm{Lin}} = \{R_{\mathrm{Lin}}, C_0, C_1\}$ where $C_a = \{a\}$ and

$$R_{\mathrm{Lin}} = \{(a, b, c) \in \{0, 1\}^3 \mid a + b = c\}$$

- $\langle \Gamma_{\mathrm{Lin}} \rangle_n$ consists of all affine subspaces of $\mathbb{Z}_2^n$

Interesting example

Interesting example (Linear systems over $\mathbb{Z}_2$)

- $A = \{0, 1\}$, $\Gamma_{\mathrm{Lin}} = \{R_{\mathrm{Lin}}, C_0, C_1\}$ where $C_a = \{a\}$ and

$$R_{\mathrm{Lin}} = \{(a, b, c) \in \{0, 1\}^3 \mid a + b = c\}$$

- $\langle \Gamma_{\mathrm{Lin}} \rangle_n$ consists of all affine subspaces of $\mathbb{Z}_2^n$
- Each subspace is a conjunction of at most n linear equations

Interesting example

Interesting example (Linear systems over $\mathbb{Z}_2$)

- $A = \{0, 1\}$, $\Gamma_{\mathrm{Lin}} = \{R_{\mathrm{Lin}}, C_0, C_1\}$ where $C_a = \{a\}$ and

$$R_{\mathrm{Lin}} = \{(a, b, c) \in \{0, 1\}^3 \mid a + b = c\}$$

- $\langle \Gamma_{\mathrm{Lin}} \rangle_n$ consists of all affine subspaces of $\mathbb{Z}_2^n$
- Each subspace is a conjunction of at most n linear equations
- Each equation can be pp-defined in $O(n)$:

Interesting example (Linear systems over $\mathbb{Z}_2$)

- $A = \{0, 1\}$, $\Gamma_{\mathrm{Lin}} = \{R_{\mathrm{Lin}}, C_0, C_1\}$ where $C_a = \{a\}$ and

$$R_{\mathrm{Lin}} = \{(a, b, c) \in \{0, 1\}^3 \mid a + b = c\}$$

- $\langle \Gamma_{\mathrm{Lin}} \rangle_n$ consists of all affine subspaces of $\mathbb{Z}_2^n$
- Each subspace is a conjunction of at most n linear equations
- Each equation can be pp-defined in $O(n)$:
  - for example, $x_1 + x_2 + x_3 = 1$ is defined by

$$(\exists u_1)(\exists u_2)(x_1 + x_2 = u_1 \ \wedge \ u_1 + x_3 = u_2 \ \wedge \ u_2 = 1)$$

Interesting example

Interesting example (Linear systems over $\mathbb{Z}_2$)

- $A = \{0,1\}$, $\Gamma_{\mathrm{Lin}} = \{R_{\mathrm{Lin}}, C_0, C_1\}$ where $C_a = \{a\}$ and

$$R_{\mathrm{Lin}} = \{(a,b,c) \in \{0,1\}^3 \mid a + b = c\}$$

- $\langle \Gamma_{\mathrm{Lin}} \rangle_n$ consists of all affine subspaces of $\mathbb{Z}_2^n$
- Each subspace is a conjunction of at most n linear equations
- Each equation can be pp-defined in $O(n)$:
  - for example, $x_1 + x_2 + x_3 = 1$ is defined by

$$(\exists u_1)(\exists u_2)(x_1 + x_2 = u_1 \;\wedge\; u_1 + x_3 = u_2 \;\wedge\; u_2 = 1)$$

  - in general, $x_{i_1} + x_{i_2} + \cdots + x_{i_k} = a$ is defined by

$$(\exists u_1)\dots(\exists u_{k-1})(\bigwedge_{1 \le j \le k-1} R_{\mathrm{Lin}}(x_{i_j}, x_{i_{j+1}}, u_j) \wedge C_a(u_k))$$

Interesting example

Interesting example (Linear systems over $\mathbb{Z}_2$)

- $A = \{0, 1\}$, $\Gamma_{\mathrm{Lin}} = \{R_{\mathrm{Lin}}, C_0, C_1\}$ where $C_a = \{a\}$ and

$$R_{\mathrm{Lin}} = \{(a, b, c) \in \{0, 1\}^3 \mid a + b = c\}$$

- $\langle \Gamma_{\mathrm{Lin}} \rangle_n$ consists of all affine subspaces of $\mathbb{Z}_2^n$
- Each subspace is a conjunction of at most n linear equations
- Each equation can be pp-defined in $O(n)$:
  - for example, $x_1 + x_2 + x_3 = 1$ is defined by

$$(\exists u_1)(\exists u_2)(x_1 + x_2 = u_1 \ \wedge \ u_1 + x_3 = u_2 \ \wedge \ u_2 = 1)$$

  - in general, $x_{i_1} + x_{i_2} + \cdots + x_{i_k} = a$ is defined by

$$(\exists u_1) \ldots (\exists u_{k-1})\Big(\bigwedge_{1 \le j \le k-1} R_{\mathrm{Lin}}(x_{i_j}, x_{i_{j+1}}, u_j) \wedge C_a(u_k)\Big)$$

$\Rightarrow$ pp-definitions of length $O(n^2)$

The conjecture and the result

Few subpowers

Few subpowers

Γ has few subpowers if $|\langle\Gamma\rangle_n| \leq 2^{p(n)}$ for some polynomial $p(n)$

Few subpowers

Γ has few subpowers if $|\langle\Gamma\rangle_n| \leq 2^{p(n)}$ for some polynomial $p(n)$

Theorem ([B]IMMVW TransAMS+SICOMP 2010)

A constraint language has $2^{O(n^k)}$ subpowers iff it is invariant under a k-edge function.

Few subpowers

Γ has few subpowers if $|\langle\Gamma\rangle_n| \leq 2^{p(n)}$ for some polynomial $p(n)$

Theorem ([B]IMMVW TransAMS+SICOMP 2010)

A constraint language has $2^{O(n^k)}$ subpowers iff it is invariant under a k-edge function. In that case, $\mathrm{CSP}(\Gamma)$ can be solved by a Gaussian-elimination-like algorithm.

Few subpowers

Γ has few subpowers if $|\langle\Gamma\rangle_n| \le 2^{p(n)}$ for some polynomial $p(n)$

Theorem ([B]IMMVW TransAMS+SICOMP 2010)

A constraint language has $2^{O(n^k)}$ subpowers iff it is invariant under a k-edge function. In that case, $\mathrm{CSP}(\Gamma)$ can be solved by a Gaussian-elimination-like algorithm. Otherwise, it has $\Omega(2^{c^n})$ subpowers for some $c > 1$.

Few subpowers

Γ has few subpowers if $|\langle\Gamma\rangle_n| \leq 2^{p(n)}$ for some polynomial $p(n)$

Theorem ([B]IMMVW TransAMS+SICOMP 2010)

A constraint language has $2^{O(n^k)}$ subpowers iff it is invariant under a k-edge function. In that case, $\mathrm{CSP}(\Gamma)$ can be solved by a Gaussian-elimination-like algorithm. Otherwise, it has $\Omega(2^{c^n})$ subpowers for some $c > 1$.

- Γ_{2SAT} is invariant under the 2-edge function called majority:

Few subpowers

Γ has few subpowers if $|\langle\Gamma\rangle_n| \leq 2^{p(n)}$ for some polynomial $p(n)$

Theorem ([B]IMMVW TransAMS+SICOMP 2010)

A constraint language has $2^{O(n^k)}$ subpowers iff it is invariant under a k-edge function. In that case, $\mathrm{CSP}(\Gamma)$ can be solved by a Gaussian-elimination-like algorithm. Otherwise, it has $\Omega(2^{c^n})$ subpowers for some $c > 1$.

- Γ_{2SAT} is invariant under the 2-edge function called majority: [2]
  $\mathrm{maj}(x, x, y) = \mathrm{maj}(x, y, x) = \mathrm{maj}(y, x, x) = x$

Few subpowers

Γ has few subpowers if $|\langle\Gamma\rangle_n| \leq 2^{p(n)}$ for some polynomial $p(n)$

Theorem ([B]IMMVW TransAMS+SICOMP 2010)

A constraint language has $2^{O(n^k)}$ subpowers iff it is invariant under a k-edge function. In that case, $\mathrm{CSP}(\Gamma)$ can be solved by a Gaussian-elimination-like algorithm. Otherwise, it has $\Omega(2^{c^n})$ subpowers for some $c > 1$.

- Γ_{2SAT} is invariant under the 2-edge function called majority: [2]
  $\mathrm{maj}(x, x, y) = \mathrm{maj}(x, y, x) = \mathrm{maj}(y, x, x) = x$

[2] In general, a k-ary function $f(x, x, \ldots, x, y) = \cdots = f(y, x, \ldots, x) = x$, called near-unanimity is equivalent to the k-Helly property (boring!)

Few subpowers

Γ has few subpowers if $|\langle\Gamma\rangle_n| \leq 2^{p(n)}$ for some polynomial $p(n)$

Theorem ([B]IMMVW TransAMS+SICOMP 2010)

A constraint language has $2^{O(n^k)}$ subpowers iff it is invariant under a k-edge function. In that case, $\mathrm{CSP}(\Gamma)$ can be solved by a Gaussian-elimination-like algorithm. Otherwise, it has $\Omega(2^{c^n})$ subpowers for some $c > 1$.

- Γ_{2SAT} is invariant under the 2-edge function called majority: [2] $\mathrm{maj}(x, x, y) = \mathrm{maj}(x, y, x) = \mathrm{maj}(y, x, x) = x$
- Γ_{Lin} is invariant under the 2-edge Mal'tsev function $x - y + z$

[2] In general, a k-ary function $f(x, x, \ldots, x, y) = \cdots = f(y, x, \ldots, x) = x$, called near-unanimity is equivalent to the k-Helly property (boring!)

Few subpowers

Γ has few subpowers if $|\langle\Gamma\rangle_n| \leq 2^{p(n)}$ for some polynomial $p(n)$

Theorem ([B]IMMVW TransAMS+SICOMP 2010)

A constraint language has $2^{O(n^k)}$ subpowers iff it is invariant under a k-edge function. In that case, $\mathrm{CSP}(\Gamma)$ can be solved by a Gaussian-elimination-like algorithm. Otherwise, it has $\Omega(2^{c^n})$ subpowers for some $c > 1$.

- Γ_{2SAT} is invariant under the 2-edge function called majority: [2] $\mathrm{maj}(x, x, y) = \mathrm{maj}(x, y, x) = \mathrm{maj}(y, x, x) = x$
- Γ_{Lin} is invariant under the 2-edge Mal'tsev function $x - y + z$

(general k-edge is a "combination" of those two types of behavior)

[2] In general, a k-ary function $f(x, x, \ldots, x, y) = \cdots = f(y, x, \ldots, x) = x$, called near-unanimity is equivalent to the k-Helly property (boring!)

Few subpowers = short definitions?

Few subpowers = short definitions?

Conjecture (B., Kompatscher)

Few subpowers = short definitions?

Conjecture (B., Kompatscher)

(weak) Γ has short definitions iff it has few subpowers.

Few subpowers = short definitions?

Conjecture (B., Kompatscher)

(weak) Γ has short definitions iff it has few subpowers.

(strong) Γ has $O(n^k)$ definitions iff it has a k-edge function.

Few subpowers = short definitions?

Conjecture (B., Kompatscher)

(weak) Γ has short definitions iff it has few subpowers.

(strong) Γ has $O(n^k)$ definitions iff it has a k-edge function.

- Short definitions imply few subpowers (cardinality argument)

Few subpowers = short definitions?

Conjecture (B., Kompatscher)

(weak) Γ has short definitions iff it has few subpowers.

(strong) Γ has $O(n^k)$ definitions iff it has a k-edge function.

- Short definitions imply few subpowers (cardinality argument)
- True for $|A| = \{0, 1\}$:

Few subpowers = short definitions?

Conjecture (B., Kompatscher)

(weak) Γ has short definitions iff it has few subpowers.

(strong) Γ has $O(n^k)$ definitions iff it has a k-edge function.

- Short definitions imply few subpowers (cardinality argument)
- True for $|A| = \{0,1\}$: essentially only Γ_{2SAT} and Γ_{Lin} (Post's lattice 1941, first noted by Lagerkvist, Wahlström 2014)

Few subpowers = short definitions?

Conjecture (B., Kompatscher)

(weak) Γ has short definitions iff it has few subpowers.

(strong) Γ has $O(n^k)$ definitions iff it has a k-edge function.

- Short definitions imply few subpowers (cardinality argument)
- True for $|A| = \{0, 1\}$: essentially only Γ_{2SAT} and Γ_{Lin} (Post's lattice 1941, first noted by Lagerkvist, Wahlström 2014)
- True if invariant under a near-unanimity (Helly property)

Few subpowers = short definitions?

Conjecture (B., Kompatscher)

(weak) Γ has short definitions iff it has few subpowers.

(strong) Γ has $O(n^k)$ definitions iff it has a k-edge function.

- Short definitions imply few subpowers (cardinality argument)
- True for $|A| = \{0, 1\}$: essentially only Γ_{2SAT} and Γ_{Lin} (Post's lattice 1941, first noted by Lagerkvist, Wahlström 2014)
- True if invariant under a near-unanimity (Helly property)

Main theorem (B., Kompatscher)

True if the algebra of polymorphisms of Γ generates a residually finite variety.

Few subpowers = short definitions?

Conjecture (B., Kompatscher)

(weak) Γ has short definitions iff it has few subpowers.

(strong) Γ has $O(n^k)$ definitions iff it has a k-edge function.

- Short definitions imply few subpowers (cardinality argument)
- True for $|A| = \{0,1\}$: essentially only Γ_{2SAT} and Γ_{Lin} (Post's lattice 1941, first noted by Lagerkvist, Wahlström 2014)
- True if invariant under a near-unanimity (Helly property)

Main theorem (B., Kompatscher)

True if the algebra of polymorphisms of Γ generates a residually finite variety.[3]

[3] For groups, this means being an A-group (Sylow subgroups are abelian)

Few subpowers = short definitions?

Conjecture (B., Kompatscher)

(weak) Γ has short definitions iff it has few subpowers.

(strong) Γ has $O(n^k)$ definitions iff it has a k-edge function.

- Short definitions imply few subpowers (cardinality argument)
- True for $|A| = \{0, 1\}$: essentially only Γ_{2SAT} and Γ_{Lin} (Post's lattice 1941, first noted by Lagerkvist, Wahlström 2014)
- True if invariant under a near-unanimity (Helly property)

Main theorem (B., Kompatscher)

True if the algebra of polymorphisms of Γ generates a residually finite variety.[3] **Corollary** True if $|A| = 3$.

[3] For groups, this means being an A-group (Sylow subgroups are abelian)

(I don't have time for) the proof

Proof overview

Proof overview

I. Switch to the right formalism (algebras, multisorted)

Proof overview

I. Switch to the right formalism (algebras, multisorted)

II. Get rid of the boring case (reduce to parallelogram relations)

Proof overview

I. Switch to the right formalism (algebras, multisorted)
II. Get rid of the boring case (reduce to parallelogram relations)
III. Reduce to “equation-like” relations (critical, reduced)

Proof overview

I. Switch to the right formalism (algebras, multisorted)

II. Get rid of the boring case (reduce to parallelogram relations)

III. Reduce to “equation-like” relations (critical, reduced)

IV. Simlulate the “shortening” construction for linear equations

Proof overview

I. Switch to the right formalism (algebras, multisorted)

II. Get rid of the boring case (reduce to parallelogram relations)

III. Reduce to "equation-like" relations (critical, reduced)

IV. Simlulate the "shortening" construction for linear equations[4]

---

[4]Step IV. is the only place where we need residual finiteness. Otherwise, in "$x + y = u$" the domain for u may grow too fast (in general, "$x + y \neq y + x$").

Step I – Switch to the right formalism: algebras

Step I – Switch to the right formalism: algebras

$R \subseteq A^n$ is invariant under $f : A^k \to A$, write $R \perp f$:

$$\mathbf{a^i} \in R \text{ for } 1 \leq i \leq k \;\Rightarrow\; f(\mathbf{a^1}, \ldots, \mathbf{a^k}) \in R$$

Step I – Switch to the right formalism: algebras

$R \subseteq A^n$ is invariant under $f : A^k \to A$, write $R \perp f$:

$$\mathbf{a^i} \in R \text{ for } 1 \leq i \leq k \;\Rightarrow\; f(\mathbf{a^1}, \ldots, \mathbf{a^k}) \in R$$

Fact: $\langle \Gamma \rangle = (\Gamma^\perp)^\perp$,

Step I – Switch to the right formalism: algebras

$R \subseteq A^n$ is invariant under $f : A^k \to A$, write $R \perp f$:

$$\mathbf{a^i} \in R \text{ for } 1 \leq i \leq k \;\Rightarrow\; f(\mathbf{a^1}, \ldots, \mathbf{a^k}) \in R$$

Fact: $\langle \Gamma \rangle = (\Gamma^{\perp})^{\perp}$, and also $\langle \mathcal{F} \rangle = (\mathcal{F}^{\perp})^{\perp}$ (the *function clone*, i.e. all term functions built from $\mathcal{F}$)

Step I – Switch to the right formalism: algebras

$R \subseteq A^n$ is invariant under $f : A^k \to A$, write $R \perp f$:

$$\mathbf{a^i} \in R \text{ for } 1 \leq i \leq k \;\Rightarrow\; f(\mathbf{a^1}, \ldots, \mathbf{a^k}) \in R$$

Fact: $\langle \Gamma \rangle = (\Gamma^\perp)^\perp$, and also $\langle \mathcal{F} \rangle = (\mathcal{F}^\perp)^\perp$ (the *function clone*, i.e. all term functions built from $\mathcal{F}$)

Examples

Step I – Switch to the right formalism: algebras

$R \subseteq A^n$ is invariant under $f : A^k \to A$, write $R \perp f$:

$$\mathbf{a^i} \in R \text{ for } 1 \leq i \leq k \;\Rightarrow\; f(\mathbf{a^1}, \ldots, \mathbf{a^k}) \in R$$

Fact: $\langle \Gamma \rangle = (\Gamma^{\perp})^{\perp}$, and also $\langle \mathcal{F} \rangle = (\mathcal{F}^{\perp})^{\perp}$ (the *function clone*, i.e. all term functions built from $\mathcal{F}$)

Examples

- $\langle \Gamma_{\mathrm{2SAT}} \rangle = \{\mathrm{maj}\}^{\perp}$

Step I – Switch to the right formalism: algebras

$R \subseteq A^n$ is invariant under $f : A^k \to A$, write $R \perp f$:

$$\mathbf{a^i} \in R \text{ for } 1 \leq i \leq k \;\Rightarrow\; f(\mathbf{a^1}, \ldots, \mathbf{a^k}) \in R$$

Fact: $\langle \Gamma \rangle = (\Gamma^\perp)^\perp$, and also $\langle \mathcal{F} \rangle = (\mathcal{F}^\perp)^\perp$ (the *function clone*, i.e. all term functions built from $\mathcal{F}$)

Examples

- $\langle \Gamma_{\mathrm{2SAT}} \rangle = \{\mathrm{maj}\}^\perp$
- $\langle \Gamma_{\mathrm{Lin}} \rangle = \{x - y + z\}^\perp$

Step I – Switch to the right formalism: algebras

$R \subseteq A^n$ is invariant under $f : A^k \to A$, write $R \perp f$:

$$\mathbf{a^i} \in R \text{ for } 1 \leq i \leq k \;\Rightarrow\; f(\mathbf{a^1}, \ldots, \mathbf{a^k}) \in R$$

Fact: $\langle \Gamma \rangle = (\Gamma^\perp)^\perp$, and also $\langle \mathcal{F} \rangle = (\mathcal{F}^\perp)^\perp$ (the *function clone*, i.e. all term functions built from $\mathcal{F}$)

Examples

- $\langle \Gamma_{\mathrm{2SAT}} \rangle = \{\mathrm{maj}\}^\perp$
- $\langle \Gamma_{\mathrm{Lin}} \rangle = \{x - y + z\}^\perp$
- $\{\leq\}^\perp$ = all monotone Boolean functions

Step I – Switch to the right formalism: algebras

$R \subseteq A^n$ is invariant under $f : A^k \to A$, write $R \perp f$:

$$\mathbf{a^i} \in R \text{ for } 1 \leq i \leq k \;\Rightarrow\; f(\mathbf{a^1}, \ldots, \mathbf{a^k}) \in R$$

Fact: $\langle \Gamma \rangle = (\Gamma^{\perp})^{\perp}$, and also $\langle \mathcal{F} \rangle = (\mathcal{F}^{\perp})^{\perp}$ (the *function clone*, i.e. all term functions built from $\mathcal{F}$)

Examples

- $\langle \Gamma_{\mathrm{2SAT}} \rangle = \{\mathrm{maj}\}^{\perp}$
- $\langle \Gamma_{\mathrm{Lin}} \rangle = \{x - y + z\}^{\perp}$
- $\{\leq\}^{\perp}$ = all monotone Boolean functions

Observe: If $\langle \Gamma \rangle = \langle \Gamma' \rangle$, then Γ has short definitions iff Γ' does.

Step I – Switch to the right formalism: algebras

$R \subseteq A^n$ is invariant under $f : A^k \to A$, write $R \perp f$:

$$\mathbf{a^i} \in R \text{ for } 1 \leq i \leq k \;\Rightarrow\; f(\mathbf{a^1}, \ldots, \mathbf{a^k}) \in R$$

Fact: $\langle \Gamma \rangle = (\Gamma^\perp)^\perp$, and also $\langle \mathcal{F} \rangle = (\mathcal{F}^\perp)^\perp$ (the *function clone*, i.e. all term functions built from $\mathcal{F}$)

Examples

- $\langle \Gamma_{\mathrm{2SAT}} \rangle = \{\mathrm{maj}\}^\perp$
- $\langle \Gamma_{\mathrm{Lin}} \rangle = \{x - y + z\}^\perp$
- $\{\leq\}^\perp$ = all monotone Boolean functions

Observe: If $\langle \Gamma \rangle = \langle \Gamma' \rangle$, then Γ has short definitions iff Γ' does.

Thus natural to consider the polymorphism algebra $\mathbf{A} = (A; \Gamma^\perp)$.

Step I – Switch to the right formalism: algebras

$R \subseteq A^n$ is invariant under $f : A^k \to A$, write $R \perp f$:

$$\mathbf{a^i} \in R \text{ for } 1 \leq i \leq k \;\Rightarrow\; f(\mathbf{a^1}, \ldots, \mathbf{a^k}) \in R$$

Fact: $\langle \Gamma \rangle = (\Gamma^{\perp})^{\perp}$, and also $\langle \mathcal{F} \rangle = (\mathcal{F}^{\perp})^{\perp}$ (the *function clone*, i.e. all term functions built from $\mathcal{F}$)

Examples

- $\langle \Gamma_{\mathrm{2SAT}} \rangle = \{\mathrm{maj}\}^{\perp}$
- $\langle \Gamma_{\mathrm{Lin}} \rangle = \{x - y + z\}^{\perp}$
- $\{\leq\}^{\perp}$ = all monotone Boolean functions

Observe: If $\langle \Gamma \rangle = \langle \Gamma' \rangle$, then Γ has short definitions iff Γ' does.

Thus natural to consider the polymorphism algebra $\mathbf{A} = (A; \Gamma^{\perp})$.
Invariant relations are sub-[universes of]powers of $\mathbf{A}$, $R \leq \mathbf{A}^n$.

Step I – Switch to the right formalism: multisorted

Step I – Switch to the right formalism: multisorted

Fundamental theorem of. . .

Step I – Switch to the right formalism: multisorted

Fundamental theorem of...

- arithmetic: $n = p_1^{e_1} \cdots p_k^{e_k}$ e.g. $6 = 2 \cdot 3$

Step I – Switch to the right formalism: multisorted

Fundamental theorem of...

- arithmetic: $n = p_1^{e_1} \cdots p_k^{e_k}$ e.g. $6 = 2 \cdot 3$
- abelian groups: $G = \mathbb{Z}_{p_1}^{e_1} \times \cdots \times \mathbb{Z}_{p_k}^{e_k}$ e.g. $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$

Step I – Switch to the right formalism: multisorted

Fundamental theorem of...

- arithmetic: $n = p_1^{e_1} \cdots p_k^{e_k}$ e.g. $6 = 2 \cdot 3$
- abelian groups: $G = \mathbb{Z}_{p_1}^{e_1} \times \cdots \times \mathbb{Z}_{p_k}^{e_k}$ e.g. $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$
- general algebras: $\mathbf{A} \leq \mathbf{A_1} \times \cdots \times \mathbf{A_k}$ where $\mathbf{A_i} \in \mathrm{HSP}(\mathbf{A})$ are subdirectly irreducible (SI)

Step I – Switch to the right formalism: multisorted

Fundamental theorem of...

- arithmetic: $n = p_1^{e_1} \cdots p_k^{e_k}$ e.g. $6 = 2 \cdot 3$
- abelian groups: $G = \mathbb{Z}_{p_1}^{e_1} \times \cdots \times \mathbb{Z}_{p_k}^{e_k}$ e.g. $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$
- general algebras: $\mathbf{A} \leq \mathbf{A_1} \times \cdots \times \mathbf{A_k}$ where $\mathbf{A_i} \in \mathrm{HSP}(\mathbf{A})$ are subdirectly irreducible (SI)

Working with subdirect decompositions...

Step I – Switch to the right formalism: multisorted

Fundamental theorem of...

- arithmetic: $n = p_1^{e_1} \cdots p_k^{e_k}$ e.g. $6 = 2 \cdot 3$
- abelian groups: $G = \mathbb{Z}_{p_1}^{e_1} \times \cdots \times \mathbb{Z}_{p_k}^{e_k}$ e.g. $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$
- general algebras: $\mathbf{A} \leq \mathbf{A_1} \times \cdots \times \mathbf{A_k}$ where $\mathbf{A_i} \in \mathrm{HSP}(\mathbf{A})$ are subdirectly irreducible (SI)

Working with subdirect decompositions...

- Residually finite = finite bound on SIs = $\exists N$ all $\mathbf{A_i} \in \mathrm{HS}(\mathbf{A}^N)$

Step I – Switch to the right formalism: multisorted

Fundamental theorem of...

- arithmetic: $n = p_1^{e_1} \cdots p_k^{e_k}$ e.g. $6 = 2 \cdot 3$
- abelian groups: $G = \mathbb{Z}_{p_1}^{e_1} \times \cdots \times \mathbb{Z}_{p_k}^{e_k}$ e.g. $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$
- general algebras: $\mathbf{A} \leq \mathbf{A_1} \times \cdots \times \mathbf{A_k}$ where $\mathbf{A_i} \in \mathrm{HSP}(\mathbf{A})$ are subdirectly irreducible (SI)

Working with subdirect decompositions...

- Residually finite = finite bound on SIs = $\exists N$ all $\mathbf{A_i} \in \mathrm{HS}(\mathbf{A}^N)$
- Multisorted relations: $R \leq A^n \longleftrightarrow R' \leq \prod_{j=1}^m \mathbf{A_{i_j}}$

Step I – Switch to the right formalism: multisorted

Fundamental theorem of...

- arithmetic: $n = p_1^{e_1} \cdots p_k^{e_k}$ e.g. $6 = 2 \cdot 3$
- abelian groups: $G = \mathbb{Z}_{p_1}^{e_1} \times \cdots \times \mathbb{Z}_{p_k}^{e_k}$ e.g. $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$
- general algebras: $\mathbf{A} \leq \mathbf{A_1} \times \cdots \times \mathbf{A_k}$ where $\mathbf{A_i} \in \mathrm{HSP}(\mathbf{A})$ are subdirectly irreducible (SI)

Working with subdirect decompositions...

- Residually finite = finite bound on SIs = $\exists N$ all $\mathbf{A_i} \in \mathrm{HS}(\mathbf{A}^N)$
- Multisorted relations: $R \leq A^n \longleftrightarrow R' \leq \prod_{j=1}^{m} \mathbf{A_{i_j}}$
- Multisorted definitions over a family of algebras $\{\mathbf{A_1}, \ldots \mathbf{A_k}\}$

Step I – Switch to the right formalism: multisorted

Fundamental theorem of...

- arithmetic: $n = p_1^{e_1} \cdots p_k^{e_k}$ e.g. $6 = 2 \cdot 3$
- abelian groups: $G = \mathbb{Z}_{p_1}^{e_1} \times \cdots \times \mathbb{Z}_{p_k}^{e_k}$ e.g. $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$
- general algebras: $\mathbf{A} \leq \mathbf{A_1} \times \cdots \times \mathbf{A_k}$ where $\mathbf{A_i} \in \mathrm{HSP}(\mathbf{A})$ are subdirectly irreducible (SI)

Working with subdirect decompositions...

- Residually finite = finite bound on SIs = $\exists N$ all $\mathbf{A_i} \in \mathrm{HS}(\mathbf{A}^N)$
- Multisorted relations: $R \leq A^n \longleftrightarrow R' \leq \prod_{j=1}^{m} \mathbf{A_{i_j}}$
- Multisorted definitions over a family of algebras $\{\mathbf{A_1}, \ldots \mathbf{A_k}\}$
- $\mathbf{A}$ has pp-definitions of length $O(n^k)$ iff $\{\mathbf{A_1}, \ldots \mathbf{A_k}\}$ does, etc. (some technical work needed here)

Step II – Get rid of the boring case

Step II – Get rid of the boring case

Lemma (Kearnes, Szendrei 2012 + Brady 2022)

If Γ *is invariant under a* k*-edge function, then every* $R \in \langle \Gamma \rangle$ *can be written as*

$$R = R' \wedge \bigwedge_{|I| \leq k} \mathrm{proj}_I(R)$$

Step II – Get rid of the boring case

Lemma (Kearnes, Szendrei 2012 + Brady 2022)

If Γ is invariant under a k-edge function, then every $R \in \langle\Gamma\rangle$ can be written as

$$R = R' \wedge \bigwedge_{|I| \leq k} \mathrm{proj}_I(R)$$

for some $R' \in \langle\Gamma\rangle$ with the parallelogram property:

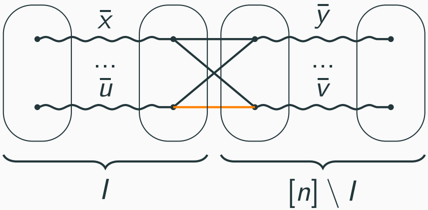
Step II – Get rid of the boring case

Lemma (Kearnes, Szendrei 2012 + Brady 2022)

If Γ is invariant under a k-edge function, then every $R \in \langle \Gamma \rangle$ can be written as

$$R = R' \wedge \bigwedge_{|I| \leq k} \mathrm{proj}_I(R)$$

for some $R' \in \langle \Gamma \rangle$ with the parallelogram property:



For every $I \subset [n]$:

$(\bar{x}, \bar{y}), (\bar{x}, \bar{v}), (\bar{u}, \bar{y}) \in R'$
$\Rightarrow (\bar{u}, \bar{v}) \in R'$

[Picture by Michael]

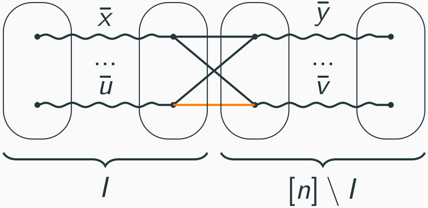
Step II – Get rid of the boring case

Lemma (Kearnes, Szendrei 2012 + Brady 2022)

If Γ is invariant under a k-edge function, then every $R \in \langle \Gamma \rangle$ can be written as

$$R = R' \wedge \bigwedge_{|I| \leq k} \mathrm{proj}_I(R)$$

for some $R' \in \langle \Gamma \rangle$ with the parallelogram property:



For every $I \subset [n]$:

$(\bar{x}, \bar{y}), (\bar{x}, \bar{v}), (\bar{u}, \bar{y}) \in R'$

$\Rightarrow (\bar{u}, \bar{v}) \in R'$

[Picture by Michael]

Examples

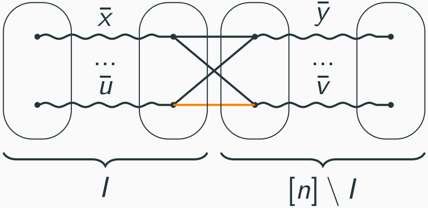
Step II – Get rid of the boring case

Lemma (Kearnes, Szendrei 2012 + Brady 2022)

If Γ is invariant under a k-edge function, then every $R \in \langle \Gamma \rangle$ can be written as

$$R = R' \wedge \bigwedge_{|I| \leq k} \mathrm{proj}_I(R)$$

for some $R' \in \langle \Gamma \rangle$ with the parallelogram property:



For every $I \subset [n]$:

$(\bar{x}, \bar{y}), (\bar{x}, \bar{v}), (\bar{u}, \bar{y}) \in R'$

$\Rightarrow (\bar{u}, \bar{v}) \in R'$

[Picture by Michael]

Examples

- Γ_{Lin}: $R' = R$ (affine subspaces have the parallelogram property)

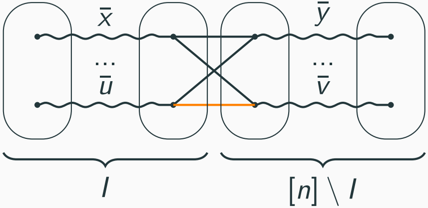
Step II – Get rid of the boring case

Lemma (Kearnes, Szendrei 2012 + Brady 2022)

If Γ is invariant under a k-edge function, then every $R \in \langle\Gamma\rangle$ can be written as

$$R = R' \wedge \bigwedge_{|I|\leq k} \mathrm{proj}_I(R)$$

for some $R' \in \langle\Gamma\rangle$ with the parallelogram property:



For every $I \subset [n]$:

$(\bar{x}, \bar{y}), (\bar{x}, \bar{v}), (\bar{u}, \bar{y}) \in R'$
$\Rightarrow (\bar{u}, \bar{v}) \in R'$

[Picture by Michael]

Examples

- Γ_{Lin}: $R' = R$ (affine subspaces have the parallelogram property)
- Γ_{2SAT}: $R' = A^n$, already $R = \bigwedge_{|I|\leq 2} \mathrm{proj}_I(R)$ (boring!)

Step III – Reduce to “equation-like” relations

Step III – Reduce to "equation-like" relations

$R \in \langle \Gamma \rangle$ is critical if it is $\wedge$-*irreducible* and has *no dummy variables*

Step III – Reduce to "equation-like" relations

$R \in \langle \Gamma \rangle$ is critical if it is $\wedge$-*irreducible* and has *no dummy variables*

Lemma: Every parallelogram relation is an intersection of at most $n \cdot |A|^2$ critical parallelogram relations (c.p.r.'s).

Step III – Reduce to “equation-like” relations

$R \in \langle \Gamma \rangle$ is critical if it is $\wedge$-*irreducible* and has *no dummy variables*

Lemma: Every parallelogram relation is an intersection of at most $n \cdot |A|^2$ critical parallelogram relations (c.p.r.'s).

Proof: somewhat like choosing codimension-many linear equations to define a subspace

Step III – Reduce to "equation-like" relations

$R \in \langle \Gamma \rangle$ is critical if it is $\wedge$-*irreducible* and has *no dummy variables*

Lemma: Every parallelogram relation is an intersection of at most $n \cdot |A|^2$ critical parallelogram relations (c.p.r.'s).

Proof: somewhat like choosing codimension-many linear equations to define a subspace

Similarity "$x_1 + x_2 = x'_1 + x'_2$ iff for some u, $x_1 + x_2 = u$ and $x'_1 + x'_2 = u$"

Step III – Reduce to "equation-like" relations

$R \in \langle \Gamma \rangle$ is critical if it is $\wedge$-*irreducible* and has *no dummy variables*

Lemma: Every parallelogram relation is an intersection of at most $n \cdot |A|^2$ critical parallelogram relations (c.p.r.'s).

Proof: somewhat like choosing codimension-many linear equations to define a subspace

Similarity "$x_1 + x_2 = x_1' + x_2'$ iff for some u, $x_1 + x_2 = u$ and $x_1' + x_2' = u$"

The linkedness congruence $\sim_I$ on $\mathrm{proj}_I R$:

$$\mathbf{x} \sim_I \mathbf{x}' \;\text{ iff }\; (\exists \mathbf{z})(R(\mathbf{x}, \mathbf{z}) \wedge R(\mathbf{x}', \mathbf{z}))$$

Step III – Reduce to "equation-like" relations

$R \in \langle \Gamma \rangle$ is critical if it is $\wedge$-*irreducible* and has *no dummy variables*

Lemma: Every parallelogram relation is an intersection of at most $n \cdot |A|^2$ critical parallelogram relations (c.p.r.'s).

Proof: somewhat like choosing codimension-many linear equations to define a subspace

Similarity "$x_1 + x_2 = x_1' + x_2'$ iff for some u, $x_1 + x_2 = u$ and $x_1' + x_2' = u$"

The linkedness congruence $\sim_I$ on $\mathrm{proj}_I\, R$:

$$\mathbf{x} \sim_I \mathbf{x}' \ \text{ iff } \ (\exists \mathbf{z})(R(\mathbf{x}, \mathbf{z}) \wedge R(\mathbf{x}', \mathbf{z}))$$

R is reduced if $\sim_{\{i\}}$ is trivial for any $i \in [n]$.

Step III – Reduce to "equation-like" relations

$R \in \langle \Gamma \rangle$ is critical if it is $\wedge$-*irreducible* and has *no dummy variables*

Lemma: Every parallelogram relation is an intersection of at most $n \cdot |A|^2$ critical parallelogram relations (c.p.r.'s).

Proof: somewhat like choosing codimension-many linear equations to define a subspace

Similarity "$x_1 + x_2 = x_1' + x_2'$ iff for some u, $x_1 + x_2 = u$ and $x_1' + x_2' = u$"

The linkedness congruence $\sim_I$ on $\mathrm{proj}_I R$:

$$\mathbf{x} \sim_I \mathbf{x}' \text{ iff } (\exists \mathbf{z})(R(\mathbf{x}, \mathbf{z}) \wedge R(\mathbf{x}', \mathbf{z}))$$

R is reduced if $\sim_{\{i\}}$ is trivial for any $i \in [n]$.

Easy: C.p.r.'s can be defined from reduced c.p.r.'s in $O(n)$

Step III – Reduce to "equation-like" relations

$R \in \langle \Gamma \rangle$ is critical if it is $\wedge$-*irreducible* and has *no dummy variables*

Lemma: Every parallelogram relation is an intersection of at most $n \cdot |A|^2$ critical parallelogram relations (c.p.r.'s).

Proof: somewhat like choosing codimension-many linear equations to define a subspace

Similarity "$x_1 + x_2 = x_1' + x_2'$ iff for some u, $x_1 + x_2 = u$ and $x_1' + x_2' = u$"

The linkedness congruence $\sim_I$ on $\mathrm{proj}_I R$:

$$\mathbf{x} \sim_I \mathbf{x}' \quad \text{iff} \quad (\exists \mathbf{z})(R(\mathbf{x}, \mathbf{z}) \wedge R(\mathbf{x}', \mathbf{z}))$$

R is reduced if $\sim_{\{i\}}$ is trivial for any $i \in [n]$.

Easy: C.p.r.'s can be defined from reduced c.p.r.'s in $O(n)$

Key Lemma: If R is a reduced c.p.r., then for any $I \subset [n]$ the algebra $\mathbf{A_I} = {}^{\mathrm{proj}_I R}/_{\sim_I}$ is SI. (multisorted Kearnes, Szendrei)

Step IV – Simlulate “shortening” linear equations

Step IV – Simlulate “shortening” linear equations

$\Gamma' =$ all multisorted 3-ary relations over $\mathrm{HS}(\mathbf{A}^N)$.

Step IV – Simlulate “shortening” linear equations

$\Gamma' =$ all multisorted 3-ary relations over $\mathrm{HS}(\mathbf{A}^N)$. By induction on n: a reduced c.p.r. $R \in \langle \Gamma' \rangle$ has a $O(n)$-long pp-definition.

Step IV – Simlulate "shortening" linear equations

$\Gamma' =$ all multisorted 3-ary relations over $\mathrm{HS}(\mathbf{A}^N)$. By induction on n: a reduced c.p.r. $R \in \langle \Gamma' \rangle$ has a $O(n)$-long pp-definition.

Define:

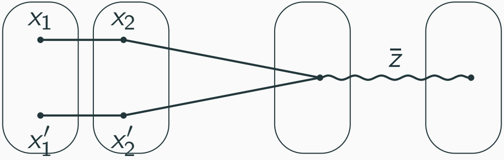
$$R(x_1, \ldots, x_n) \leftrightarrow (\exists u \in \mathbf{A_{12}})(Q(x_1, x_2, u) \wedge R'(u, x_3, \ldots, x_n))$$

Step IV – Simlulate “shortening” linear equations

$\Gamma' =$ all multisorted 3-ary relations over $\mathrm{HS}(\mathbf{A}^N)$. By induction on n: a reduced c.p.r. $R \in \langle \Gamma' \rangle$ has a $O(n)$-long pp-definition.

Define:

$$R(x_1, \ldots, x_n) \leftrightarrow (\exists u \in \mathbf{A_{12}})(Q(x_1, x_2, u) \wedge R'(u, x_3, \ldots, x_n))$$



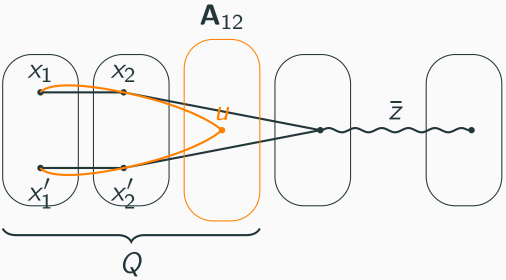
[Picture by Michael]

Step IV – Simlulate "shortening" linear equations

$\Gamma' =$ all multisorted 3-ary relations over $\mathrm{HS}(\mathbf{A}^N)$. By induction on n: a reduced c.p.r. $R \in \langle \Gamma' \rangle$ has a $O(n)$-long pp-definition.

Define:

$$R(x_1, \ldots, x_n) \leftrightarrow (\exists u \in \mathbf{A_{12}})(Q(x_1, x_2, u) \wedge R'(u, x_3, \ldots, x_n))$$



$(x_1, x_2, u) \in Q \Leftrightarrow$
$u = (x_1, x_2)/_\sim$

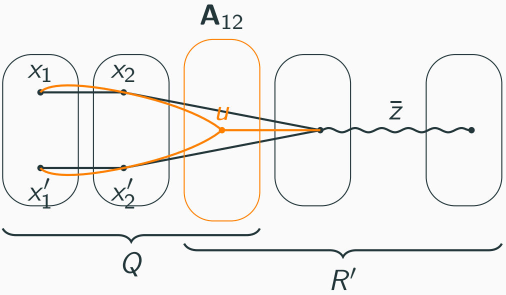
[Picture by Michael]

Step IV – Simlulate “shortening” linear equations

Γ' = all multisorted 3-ary relations over $\mathrm{HS}(\mathbf{A}^N)$. By induction on n: a reduced c.p.r. $R \in \langle \Gamma' \rangle$ has a $O(n)$-long pp-definition.

Define:

$$R(x_1, \ldots, x_n) \leftrightarrow (\exists u \in \mathbf{A_{12}})(Q(x_1, x_2, u) \wedge R'(u, x_3, \ldots, x_n))$$



$(x_1, x_2, u) \in Q \Leftrightarrow$
$u = (x_1, x_2)/_\sim$

$(y, \bar{z}) \in R' :\Leftrightarrow$
$u = (x_1, x_2)/_\sim, (x_1, x_2, \bar{z}) \in R$

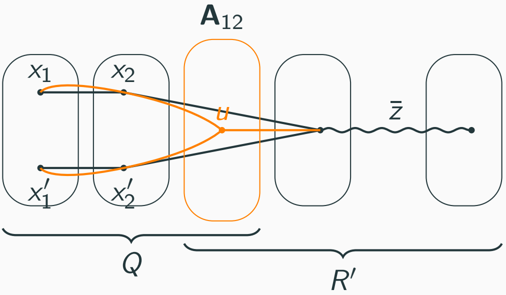
[Picture by Michael]

Step IV – Simlulate "shortening" linear equations

$\Gamma' =$ all multisorted 3-ary relations over $\mathrm{HS}(\mathbf{A}^N)$. By induction on n: a reduced c.p.r. $R \in \langle \Gamma' \rangle$ has a $O(n)$-long pp-definition.

Define:

$$R(x_1, \ldots, x_n) \leftrightarrow (\exists u \in \mathbf{A_{12}})(Q(x_1, x_2, u) \wedge R'(u, x_3, \ldots, x_n))$$



$(x_1, x_2, u) \in Q \Leftrightarrow$
$u = (x_1, x_2)/_\sim$

$(y, \bar{z}) \in R' :\Leftrightarrow$
$u = (x_1, x_2)/_\sim, (x_1, x_2, \bar{z}) \in R$

[Picture by Michael]

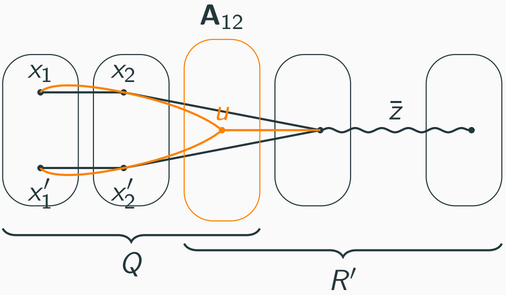
[slightly ruined by B.]

Step IV – Simlulate "shortening" linear equations

$\Gamma' =$ all multisorted 3-ary relations over $\mathrm{HS}(\mathbf{A}^N)$. By induction on n: a reduced c.p.r. $R \in \langle \Gamma' \rangle$ has a $O(n)$-long pp-definition.

Define:

$$R(x_1, \ldots, x_n) \leftrightarrow (\exists u \in \mathbf{A_{12}})(Q(x_1, x_2, u) \wedge R'(u, x_3, \ldots, x_n))$$



$(x_1, x_2, u) \in Q \Leftrightarrow$
$u = (x_1, x_2)/_\sim$

$(y, \bar{z}) \in R' :\Leftrightarrow$
$u = (x_1, x_2)/_\sim, (x_1, x_2, \bar{z}) \in R$

[Picture by Michael]
[slightly ruined by B.]

By Key Lemma, $\mathbf{A_{12}} = {}^{\mathrm{proj}_{12}\,R}/_{\sim_{12}}$ is SI, so by residual finiteness it is in $\mathrm{HS}(\mathbf{A}^N)$. Thus $Q \in \Gamma'$; the arity of R' is $n - 1$.

The application

Representing relations

Representing relations

A "representation" of $R \in \langle \Gamma \rangle$ must be both small and efficient

Representing relations

A "representation" of $R \in \langle\Gamma\rangle$ must be both small and efficient

Examples

Representing relations

A "representation" of $R \in \langle \Gamma \rangle$ must be both small and efficient

Examples

- basis of a vector subspace (+ row reduction)

Representing relations

A "representation" of $R \in \langle \Gamma \rangle$ must be both small and efficient

Examples

- basis of a vector subspace (+ row reduction)
- SGS of a permutation group (+ sifting in Schreier-Sims algo)

Representing relations

A "representation" of $R \in \langle \Gamma \rangle$ must be both small and efficient

Examples

- basis of a vector subspace (+ row reduction)
- SGS of a permutation group (+ sifting in Schreier-Sims algo)

Fact: Few subpowers $\Leftrightarrow$ small generating sets (BIMMVW 2010)

Representing relations

A "representation" of $R \in \langle \Gamma \rangle$ must be both small and efficient

Examples

- basis of a vector subspace (+ row reduction)
- SGS of a permutation group (+ sifting in Schreier-Sims algo)

Fact: Few subpowers $\Leftrightarrow$ small generating sets (BIMMVW 2010)
But are they efficient?

Representing relations

A "representation" of $R \in \langle \Gamma \rangle$ must be both small and efficient

Examples

- basis of a vector subspace (+ row reduction)
- SGS of a permutation group (+ sifting in Schreier-Sims algo)

Fact: Few subpowers $\Leftrightarrow$ small generating sets (BIMMVW 2010)
But are they efficient?

Subpower membership problem $\mathrm{SMP}(\mathbf{A})$**:**

A is a finite algebra (e.g. the polymoprhism algebra of Γ)

input tuples $\mathbf{b}, \mathbf{a^1}, \dots, \mathbf{a^k}$ from A^n

question is $\mathbf{b}$ in the subpower generated by $\mathbf{a^1}, \dots, \mathbf{a^k}$?

Representing relations

A "representation" of $R \in \langle \Gamma \rangle$ must be both small and efficient

Examples

- basis of a vector subspace (+ row reduction)
- SGS of a permutation group (+ sifting in Schreier-Sims algo)

Fact: Few subpowers $\Leftrightarrow$ small generating sets (BIMMVW 2010)

But are they efficient?

Subpower membership problem $\mathrm{SMP}(\mathbf{A})$:

$\mathbf{A}$ is a finite algebra (e.g. the polymoprhism algebra of Γ)

input tuples $\mathbf{b}, \mathbf{a^1}, \ldots, \mathbf{a^k}$ from A^n

question is $\mathbf{b}$ in the subpower generated by $\mathbf{a^1}, \ldots, \mathbf{a^k}$?

Question (BIMMVW 2010)

Is $\mathrm{SMP}(\mathbf{A})$ in P for $\mathbf{A}$ with few subpowers?

Polynomial evaluability

Polynomial evaluability

Let **A** have few subpowers

Polynomial evaluability

Let **A** have few subpowers

- **Question:** $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{P}$? (BIMMVW 2010)

Polynomial evaluability

Let **A** have few subpowers

- **Question:** $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{P}$? (BIMMVW 2010)
- **Theorem:** $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{NP}$. (Bulatov, Mayr, Szendrei 2019)

Polynomial evaluability

Let **A** have few subpowers

- **Question:** $\mathrm{SMP}(\mathbf{A})$ in P? (BIMMVW 2010)
- **Theorem:** $\mathrm{SMP}(\mathbf{A})$ in NP. (Bulatov, Mayr, Szendrei 2019)
  If **A** generates a residually small variety, then $\mathrm{SMP}(\mathbf{A})$ in P.

Polynomial evaluability

Let **A** have few subpowers

- **Question:** $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{P}$? (BIMMVW 2010)
- **Theorem:** $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{NP}$. (Bulatov, Mayr, Szendrei 2019)
  If **A** generates a residually small variety, then $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{P}$.

Fact (B., Kompatscher)

Short definitions $\Rightarrow$ $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{NP} \cap \mathtt{co\text{-}NP}$

Polynomial evaluability

Let **A** have few subpowers

- **Question:** $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{P}$? (BIMMVW 2010)
- **Theorem:** $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{NP}$. (Bulatov, Mayr, Szendrei 2019)
  If **A** generates a residually small variety, then $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{P}$.

Fact (B., Kompatscher)

Short definitions $\Rightarrow \mathrm{SMP}(\mathbf{A})$ in $\mathtt{NP} \cap \mathtt{co\text{-}NP}$

Proof: Guess $\phi(x_1, \ldots, x_n)$*, verify* $\phi(\mathbf{a^i})$ *for* $1 \leq i \leq k$ *but* $\neg\phi(\mathbf{b})$

Polynomial evaluability

Let **A** have few subpowers

- **Question:** $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{P}$? (BIMMVW 2010)
- **Theorem:** $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{NP}$. (Bulatov, Mayr, Szendrei 2019)
  If **A** generates a residually small variety, then $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{P}$.

Fact (B., Kompatscher)

Short definitions $\Rightarrow \mathrm{SMP}(\mathbf{A})$ in $\mathtt{NP} \cap \mathtt{co}\text{-}\mathtt{NP}$

Proof: Guess $\phi(x_1, \ldots, x_n)$, *verify* $\phi(\mathbf{a^i})$ *for* $1 \leq i \leq k$ *but* $\neg\phi(\mathbf{b})$

Question

Given generators for R, can we compute a short pp-definition in polynomial time?

Polynomial evaluability

Let **A** have few subpowers

- **Question:** $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{P}$? (BIMMVW 2010)
- **Theorem:** $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{NP}$. (Bulatov, Mayr, Szendrei 2019)
  If **A** generates a residually small variety, then $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{P}$.

Fact (B., Kompatscher)

Short definitions $\Rightarrow \mathrm{SMP}(\mathbf{A})$ in $\mathtt{NP} \cap \mathtt{co\text{-}NP}$

Proof: Guess $\phi(x_1, \ldots, x_n)$, *verify* $\phi(\mathbf{a^i})$ *for* $1 \leq i \leq k$ *but* $\neg\phi(\mathbf{b})$

Question

Given generators for R, can we compute a short pp-definition in polynomial time?

- If true, then $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{P}$

Polynomial evaluability

Let **A** have few subpowers

- **Question:** $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{P}$? (BIMMVW 2010)
- **Theorem:** $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{NP}$. (Bulatov, Mayr, Szendrei 2019)
  If **A** generates a residually small variety, then $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{P}$.

Fact (B., Kompatscher)

Short definitions $\Rightarrow$ $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{NP} \cap \mathtt{co}\text{-}\mathtt{NP}$

Proof: Guess $\phi(x_1, \ldots, x_n)$, *verify* $\phi(\mathbf{a^i})$ *for* $1 \leq i \leq k$ *but* $\neg\phi(\mathbf{b})$

Question

Given generators for R, can we compute a short pp-definition in polynomial time?

- If true, then $\mathrm{SMP}(\mathbf{A})$ in $\mathtt{P}$
- True for $A = \{0, 1\}$, otherwise open