Lecture 11 – TMs and grammars, Linear Bounded Automata, Intro to computability

NTIN071 Automata and Grammars

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Recap of Lecture 10

- Turing machine: two-way infinite tape, read, write, move head
- Accept iff in a final state; configurations
- TMs with output, computing a function
- Recursively enumerable vs. recursive languages (always halt).
- Construction tricks:
  - storage in state
  - multiple tracks (on a single tape)
- Variants of TMs:
  - multi-tape (independent heads),
  - nondeterministic (accept iff some choices lead to final state)

3.3 Turing Machines and grammars

Chomsky hierarchy: Type 0

$$\begin{tikzcd}
\boxed{\text{Turing machines}} \arrow[r, leftrightarrow] & \boxed{\text{Formal grammars}} \\
\boxed{\text{Linear bounded automata}} \arrow[r, leftrightarrow] & \boxed{\text{Context-sensitive grammars}} \\
\boxed{\text{Pushdown automata}} \arrow[r, leftrightarrow] & \boxed{\text{Context-free grammars}} \\
\boxed{\text{Finite automata}} \arrow[r, leftrightarrow] & \boxed{\text{Regular grammars}}
\end{tikzcd}$$

Theorem

A language is recursively enumerable, if and only if it is generated by a Type 0 grammar.

Turing machine to grammar

- First generate the relevant portion of the tape and a copy of the input word (nonterminal $\underline{X}$ for each $x \in \Gamma$, in reverse)
- Why? TM can rewrite w, G must generate it, cannot modify
- We have $wB^n\underline{W}^RQ_0B^m$, where B^n, B^m is sufficient free space
- Then simulate moves (essentially reverse configs+free space)
- In a final state erase the simulated tape, keep only w

$G = (\{S, C, D, E\} \cup \{\underline{X}\}_{x \in \Gamma} \cup \{Q_i\}_{q_i \in Q}, \Sigma, \mathcal{P}, S)$ where $\mathcal{P}$ is:

(1)	$S \to DQ_0E$	simulation starts in initial state
	$D \to xD\underline{X} \mid E$	generate input word, reverse copy for simulation
	$E \to BE \mid \epsilon$	generate sufficient free space for simulation
(2)	$\underline{X}P \to Q\underline{X'}$	for all $\delta(p, x) = (q, x', R)$ [direction reversed!]
	$\underline{X}P\underline{Y} \to \underline{X'}\,\underline{Y}Q$	for all $\delta(p, x) = (q, x', L)$
(3)	$P \to C$	for all $p \in F$
	$C\underline{X} \to C, \underline{X}C \to C$	clean the tape
	$C \to \epsilon$	finish, generated w

Example: $L = \{a^{2n} \mid n \geq 0\}$

$M = (\{q_0, q_1, q_2, q_F\}, \{a\}, \{a\}, \delta, q_0, B, \{q_F\})$ where

$$\delta(q_0, a) = (q_1, a, R),$$
$$\delta(q_1, a) = (q_0, a, R),$$
$$\delta(q_0, B) = (q_F, B, L)$$

$G = (\{S, C, D, E, Q_0, Q_1, Q_F, \underline{a}\}, \{a\}, S, \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3)$

Initialize: $\mathcal{P}_1$	**Simulate:** $\mathcal{P}_2$	**Cleanup:** $\mathcal{P}_3$
$S \to DQ_0E$	$\underline{a}Q_0 \to Q_1\underline{a}$	$Q_F \to C$
$D \to aD\underline{a} \mid E$	$\underline{a}Q_1 \to Q_0\underline{a}$	$C\underline{a} \to C$
$E \to BE \mid \epsilon$	$BQ_0\underline{a} \to B\underline{a}Q_F$	$\underline{a}C \to C$
		$BC \to C$
		$C \to \epsilon$

For $w = aa$: initialize $aaB\underline{a}\underline{a}Q_0$, simulate $aaB\underline{a}Q_F\underline{a}$, cleanup: aa

Proof

$L(M) \subseteq L(G)$

- For $w \in L(M)$ there is a finite accepting sequence of moves
- The grammar generates sufficient space
- Then we simulate the moves
- Finally clean non-input symbols

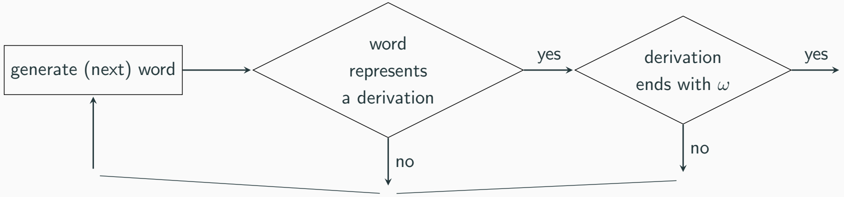
$L(G) \subseteq L(M)$

- Steps in a derivation for $w \in L(G)$ may be in different order
- But we can reorder them into the phases (1), (2), (3)
- Since we eliminated the underlined symbols, we must have generated the cleaning variable C
- In order to generate C we must have generated a final state
- A final state can only be generated from the initial state by a sequence of simulated moves $\square$

Grammar to Turing machine

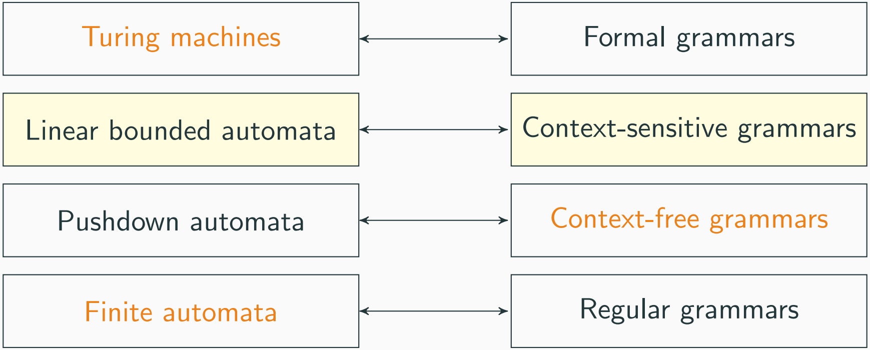
Idea: The TM sequentially generates all possible derivations.
(Note: here we do not care about efficiency.)

- code $S \Rightarrow \beta_1 \Rightarrow \ldots \Rightarrow \beta_n = \omega$ as a string $\#S\#\beta_1\#\ldots\#\omega\#$
- construct a TM accepting exactly $\#\alpha\#\beta\#$ where $\alpha \Rightarrow \beta$
- construct a TM accepting $\#\beta_1\#\ldots\#\beta_k\#$ where $\beta_1 \Rightarrow^* \beta_k$
- construct a TM generating sequentially all possible strings
- check if the string is a valid derivation ending with ω



3.4 Linear bounded automata and context-sensitive grammars

Chomsky hierarchy: Type 1



Context-sensitive languages

Theorem

The following are equivalent for a language L:

(i) *L is generated by a context-sensitive grammar.*

(ii) *L is generated by a monotone grammar.*

(iii) *L is recognized by a Linear Bounded Automaton (LBA).*

- context-sensitive grammar: $\alpha_1 A \alpha_2 \to \alpha_1 \gamma \alpha_2$ where $A \in V$, $\gamma \in (V \cup T)^+$, $\alpha_1, \alpha_2 \in (V \cup T)^*$ ($S \to \epsilon$ if S not in bodies)
- monotone grammar: $\alpha \to \beta$ where $|\alpha| \leq |\beta|$
- Linear Bounded Automaton (LBA): a nondeterministic TM only using the input portion of the tape [we formalize later]

Note: Context-sensitive grammars are monotone, $(i) \Rightarrow (ii)$ trivial. Monotone grammars do not shorten sentential forms in a derivation

Example: $L = \{a^n b^n c^n \mid n \geq 1\}$ is context-sensitive

(Recall that L is not context-free.)

A monotone grammar:

$S \to aSBC \mid abC$	*right amount of a, B, C*
$CB \to BC$	*reorder to $a^n bB^{n-1}C^n$*
$bB \to bb$	*$B \to b$ only if preceded by b*
$bC \to bc$	*$C \to c$ only if preceded by b*
$cC \to cc$	*. . . or by c*

The rule $CB \to BC$ is not context-sensitive. But we can convert it to a chain of context-sensitive rules:

$$CB \to XB,\ XB \to XY,\ XY \to BY,\ BY \to BC$$

(Same for any monotone rule, as long as there are no terminals.)

From monotone to context-sensitive grammars $(ii) \Rightarrow (i)$

Recall: separated grammar means productions of the form $\alpha \to \beta$ where either $\alpha, \beta \in V^+$ or $\alpha \in V, \beta \in T \cup \{\epsilon\}$

Lemma

Every monotone grammar can be converted to an equivalent context-sensitive grammar.

Proof: First, convert to separated grammar (as for ChNF). This preserves monotonicity, $V_a \to a$ is monotone, context-sensitive.

Then, convert every production $A_1 \dots A_m \to B_1 \dots B_n$ $(m \leq n)$ to the following chain (using new auxiliary variables C_i):

$$\begin{array}{cc}
A_1 A_2 \dots A_m \to C_1 A_2 \dots A_m & C_1 C_2 \dots C_m \to B_1 C_2 \dots C_m \\
C_1 A_2 \dots A_m \to C_1 C_2 \dots A_m & B_1 C_2 \dots C_m \to B_1 B_2 \dots C_m \\
\vdots & \vdots \\
C_1 \dots C_{m-1} A_m \to C_1 \dots C_{m-1} C_m & B_1 \dots B_{m-1} C_m \to B_1 \dots B_{m-1} B_m \dots B_n
\end{array}$$

Linear Bounded Automaton

Definition

A linear bounded automaton (LBA) is a *nondeterministic* Turing machine where the tape contains special symbols for left ($\underline{l}$) and right ($\underline{r}$) end. Those symbols cannot be rewritten and the head cannot move to the left of $\underline{l}$ or to the right of $\underline{r}$.

A word w is accepted if $q_0\underline{l}w\underline{r} \vdash^* \alpha p \beta$ for some $p \in F$

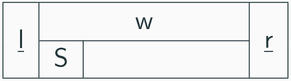
- The space for computation is given by the input word, we cannot exceed its length.
- Not a problem for context-sensitive/monotone grammars: sentential forms in a derivation cannot shorten.
- Nondeterminisim is crucial!

Construction trick: 'draw' several tape symbols into one cell (as in multi-track tape), increase space by constant factor; hence 'linear'

From context-sensitive grammar to LBA $(i) \Rightarrow (iii)$

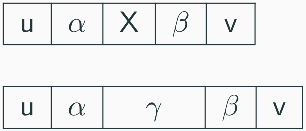
Track 1: a copy of the input w, read-only

Track 2: simulate the derivation of w



- initialize with S in first field (the rest blank)
- at the end it should contain w, compare to Track 1

- to simulate one derivation step (apply rule $\alpha X \beta \rightarrow \alpha \gamma \beta$):



- rewrite the sentential form using production rules
- nondeterministically choose which rule and where to apply it
- rewrite head to body (move the rest to the right)
- if only terminals, compare with Track 1, accept if match □

From LBA to monotone grammar

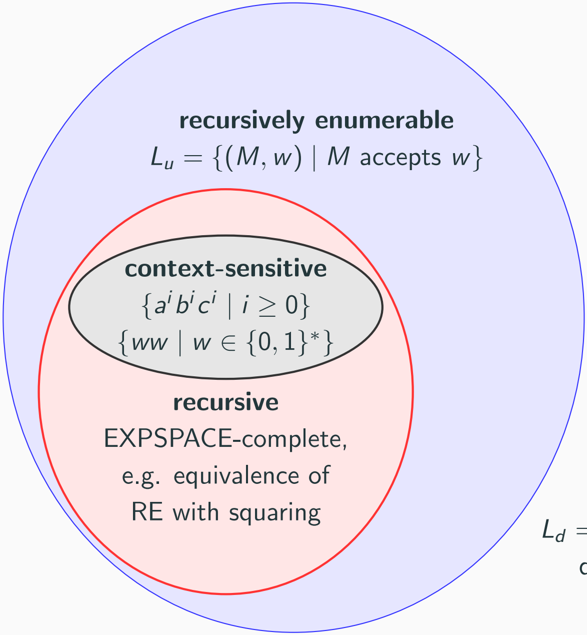
- the grammar cannot generate any ‘extra’ symbols
- we hide the computation in ‘two-track’ variables
- generate a word of the form

$$(a_0, [q_0, \underline{l}, a_0]), (a_1, a_1), \ldots, (a_n, [a_n, \underline{r}])$$

w		
$q_0, \underline{l}, a_0$		$a_n, \underline{r}$

- simulate computation in the 2nd track (as for TMs)
  - for $\delta(p, x) \ni (q, x', R)$: $P\underline{XY} \to \underline{X'}Q\underline{Y}$
  - for $\delta(p, x) \ni (q, x', L)$: $\underline{Y}P\underline{X} \to Q\underline{YX'}$
- if the state is accepting, ‘erase’ the 2nd track
- special production for generating ϵ (if $\epsilon \in L$) $\square$

Hierarchy of languages: context-sensitive and above



recursively enumerable
$L_u = \{(M, w) \mid M \text{ accepts } w\}$
context-sensitive
$\{a^i b^i c^i \mid i \geq 0\}$
$\{ww \mid w \in \{0,1\}^*\}$
recursive
EXPSPACE-complete,
e.g. equivalence of
RE with squaring

$L_d = \{w \mid \text{TM with code } w$ does not accept input $w\}$

Chapter 4: Intro to computability theory

First, a brief overview in 4 slides, without technical details

Languages and decision problems

A decision problem P: given input w (usually a 0-1 string), answer YES or NO (e.g. 'Is the given number prime?', 'Is the given picture classified as cat by the given neural net?')

$$L_P = \{w \mid P(w) \text{ answers 'YES'}\}$$

- P is (algorithmically) decidable $\Leftrightarrow$ L_P is recursive $\Leftrightarrow$ there is a TM deciding L_P (halting on every input, answering correctly)
- P is partially decidable $\Leftrightarrow$ L_P is recursively enumerable $\Leftrightarrow$ there is a TM that accepts every YES-instance w but for NO-instances it may either reject or run an infinite loop

NB: almost all problems are not even partially decidable (TMs can be represented by finite strings, so only countably many TMs)

Coming up next: a concrete example, the diagonal language

Source code for a Turing Machine & how to execute it

Source code for TMs:

- encode TMs by 01-strings, $M \rightsquigarrow \mathrm{code}(M) \in \{0,1\}^*$
- if w is not well-formed code, then say it represents a TM with no transitions, so every $w \in \{0,1\}^*$ will represent some TM
- also encode a pair of 01-strings u, v as a 01-string $\langle u, v \rangle$

The Universal language: $L_U = \{\langle \mathrm{code}(M), w \rangle \mid M \text{ accepts } w\}$

"Does a given program return true on a given input?"

Theorem

The Universal language is recursively enumerable.

Proof idea: construct the Universal Turing Machine that can simulate any TM (using its code) on any input [details later]

Barber's paradox aka the diagonal argument

The Diagonal language:

$$L_D = \{w \mid M \text{ such that } w = \mathrm{code}(M) \text{ does *not* accept } w\}$$

"Return true if the given program does not return true when fed its own source code."

Theorem

The Diagonal language is not recursively enumerable.

Proof idea: there cannot exist a TM recognizing L_D: running it on its own code would lead to Barber's paradox

"The program accepts all programs that don't accept themselves. Does the program accept itself?"

Languages that are recursively enumerable, but not recursive

Post's theorem

A language L is recursive, if and only if both L and $\overline{L}$ are recursively enumerable.

Proof idea: simulate TMs for L and $\overline{L}$ in parallel, one must halt

Corollary

The language $\overline{L_D}$ is not recursive, but it is recursively enumerable.

"Does the given program return true when fed its own code?"

Corollary

The Universal language is not recursive.

(If a TM decided L_U, we could use it to decide $\overline{L_D}$: $w \rightsquigarrow \langle w, w \rangle$)

We can execute a program, but cannot test if it runs into a loop.

Now, the technical details

Machine-readable encoding of TMs (Gödel numbering)

To encode a TM as a binary string, we first assign integers to the states, tape symbols, and directions L, R. Assume:

- the start state is always q_1, the only final state is q_2
- the first tape symbol is always 0, the second 1, the third B (other tape symbols can be assigned arbitrarily)
- the direction L is 1, the direction R is 2

Each transition $\delta(q_i, X_j) = (q_k, X_l, D_m)$ is encoded by $0^i10^j10^k10^l10^m$. Since $i, j, k, l, m \geq 1$, substring 11 doesn't occur.

The entire encoding $\mathrm{code}(M)$ consists of codes for all transitions (in any order), separated by a pair of 1's: $C_1 11 C_2 11 \ldots C_{n-1} 11 C_n$.

Similarly, we encode a tuple of 01-strings as a 01-string: separate entries by 111. We also fix an order of 01-strings, by length + lexicographically ($w_0 = \epsilon$, $w_1 = 0$, $w_2 = 1$, $w_3 = 00$, $w_4 = 01$, $\ldots$)

Example

$$M = (\{q_1, q_2, q_3\}, \{0, 1\}, \{0, 1, B\}, \delta, q_1, B, \{q_2\})$$

δ	0	1	B
$\rightarrow q_1$		$(q_3, 0, R)$	
$*q_2$			
q_3	$(q_1, 1, R)$	$(q_2, 0, R)$	$(q_3, 1, L)$.

Codes for transitions:

C_1	C_2	C_3	C_4
0100100010100	0001010100100	00010010010100	0001000100010010

The full encoding $\mathrm{code}(M)$:

01001000101001100010101001001100010010010100110001000100010010

Summary of Lecture 11

- Recursively enumerable languages are exactly those generated by (Type 0) grammars
  - TM to G: simulate moves on a reversed non-terminal copy of ω, generate sufficient space, cleanup if accepting state
  - G to TM: generate all strings, check if any of them represents a valid derivation of ω (sentential forms separated by $\#$)
- Context-sensitive languages:
  - context-sensitive grammars are equivalent to monotone grammars
  - Linear Bounded Automaton (LBA): nondeterministic TM with tape limited to the length of input
  - constructions: monotone grammar to LBA, LBA to monotone grammar
- Intro to computability: an overview
- decision problem $\leftrightsquigarrow$ the language of all 'YES' instances
- machine-readable encoding of TMs