

Lecture 4 – Regular expressions, Kleene's theorem, string substitution

NTIN071 Automata and Grammars

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** Adapted from the Czech-lecture slides by Marta Vomlelová with gratitude. The translation, some modifications, and all errors are mine.*

Recap of Lecture 3

- Nondeterministic finite automata (NFA): can 'guess' the right path to accepting, computation described by a state tree.
- ϵ -transitions: allow to change states without reading any input
- Subset construction: every NFA and ϵ -NFA is equivalent to a DFA (but can be easier to design, much smaller).
- Regular languages are closed under set operations (union, intersection, complement, difference)
- And under string operations (concatenation, iteration and positive iteration, reverse, left and right quotient)

1.8 Regular expressions

Regular expressions (RE)

- an algebraic description of languages
- declarative: express the form of the words we want to accept
- can describe all, and only, regular languages
- can be viewed as a programming language, a user/friendly description of a finite automaton

Example

- grep command in UNIX.
- Python module re
- lexical analysis, e.g. Flex (description via 'tokens' \leftrightarrow RE)

Note: syntax analysis needs a stronger tool, context-free grammars

The definition

A **regular expression** α over (finite, nonempty) Σ , $\alpha \in \text{RegE}(\Sigma)$ and the matching language $L(\alpha)$, are defined inductively:

expression	language	note
\emptyset	$L(\emptyset) = \emptyset$	empty expression
ϵ	$L(\epsilon) = \{\epsilon\}$	empty string
a	$L(\mathbf{a}) = \{a\}$	for all $a \in \Sigma$
$(\alpha + \beta)$	$L((\alpha + \beta)) = L(\alpha) \cup L(\beta)$	union (grep, re use ' ')
$(\alpha\beta)$	$L((\alpha\beta)) = L(\alpha)L(\beta)$	concatenation
α^*	$L(\alpha^*) = L(\alpha)^*$	iteration (Kleene star)

Examples, notation

Example

- The language of alternating 0s and 1s can be expressed as:
 - $(\mathbf{01})^* + (\mathbf{10})^* + \mathbf{1}(\mathbf{01})^* + \mathbf{0}(\mathbf{10})^*$
 - $(\epsilon + \mathbf{1})(\mathbf{01})^*(\epsilon + \mathbf{0})$
- $L((\mathbf{0}^*\mathbf{10}^*\mathbf{10}^*\mathbf{1})^*\mathbf{0}^*) = \{w \in \{0, 1\}^* \mid |w|_1 \equiv 0 \pmod{3}\}$

We often omit parentheses:

- priority of operators: iteration $*$ $>$ concatenation $>$ union $+$
- associativity of concatenation, union $+$
- outer parentheses

We could define, and will sometimes use, positive iteration α^+

Theorem (Kleene's theorem)

A language is regular, iff it is matched by some regular expression.

We will prove it by giving two constructions:

1. from RE to ϵ -NFA (which can be converted to a DFA)
2. from a DFA to a RE (but we could start from a ϵ -NFA)

For 2. we also mention a better algorithm: **state elimination**

RE to ϵ -NFA

By induction on the structure of α , construct a ϵ -NFA E s.t. $L(\alpha) = L(E)$ with three additional properties:

1. Exactly one accepting state.
2. No incoming edges into the initial state.
3. No outgoing edges from the accepting state.

Induction base: α is the empty string ϵ , empty set \emptyset , or a letter **a**



Induction step: $\alpha + \beta$, $\alpha\beta$, α^* (next slide)

□

RE to ϵ -NFA: Induction step

Let A, B be ϵ -NFA constructed for α, β .

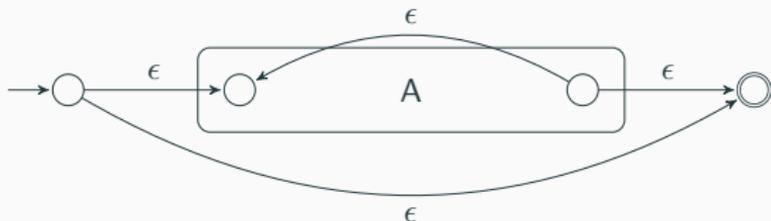
Addition $\alpha + \beta$



Concatenation $\alpha\beta$



Iteration α^*



Assume the states are $Q = \{1, \dots, n\}$ and the start state is $q_0 = 1$.

Construct a RE $R_{ij}^{(k)}$ matching words that transition from state i into state j and all intermediate states (if any) have index $\leq k$.

Then we set $\alpha = \sum_{j \in F_A} R_{1j}^{(n)}$ (from start to some accepting state)

Iteratively construct $R_{ij}^{(k)}$ for $k = 0, \dots, n$ (finite induction).

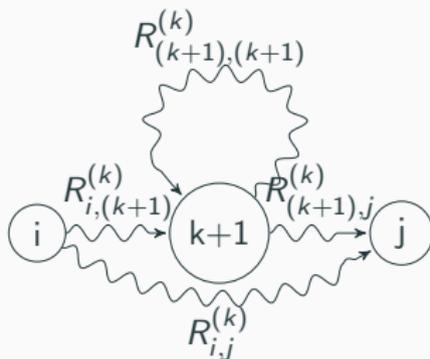
Induction base: $k = 0$

- If $i \neq j$, set $R_{ij}^{(0)} = \mathbf{a}_1 + \dots + \mathbf{a}_m$ where a_1, \dots, a_m are symbols on edges from i into j ($R_{ij}^{(0)} = \emptyset$ or $R_{ij}^{(0)} = \mathbf{a}$ for $m = 0, 1$).
- If $i = j$, $R_{ii}^{(0)} = \epsilon + \mathbf{a}_1 + \dots + \mathbf{a}_m$ where a_i 's are on loops on i .

DFA to RE: Induction step

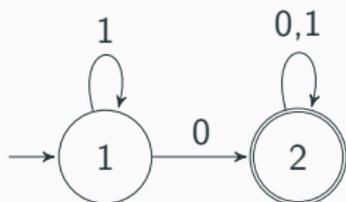
Once we have $R_{ij}^{(k)}$ for all $i, j \in Q$, we can construct $R_{ij}^{(k+1)}$:

$$R_{ij}^{(k+1)} = R_{ij}^{(k)} + R_{i,(k+1)}^{(k)} (R_{(k+1),(k+1)}^{(k)})^* R_{(k+1),j}^{(k)}$$



- paths $i \rightsquigarrow j$ not going through $k+1$: already in $R_{ij}^{(k)}$
- paths $i \rightsquigarrow j$ going through $k+1$ one or more times: $i \rightsquigarrow k+1$ (first visit), loop on $k+1$, finally (last visit) $k+1 \rightsquigarrow j$ \square

Example



Apply the construction, simplify:

$$\alpha = R_{12}^{(2)} = \mathbf{1^*0(0+1)^*}$$

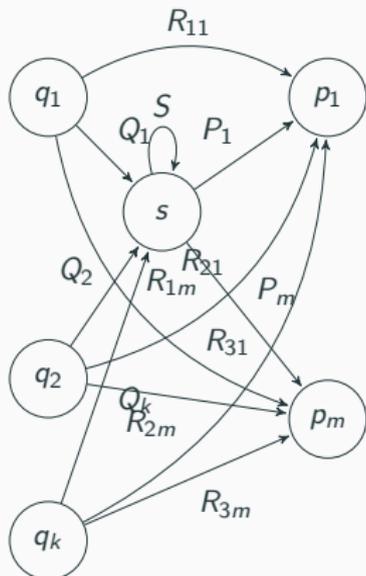
$R_{11}^{(0)}$	$\epsilon + \mathbf{1}$	$=$
$R_{12}^{(0)}$	$\mathbf{0}$	$=$
$R_{21}^{(0)}$	\emptyset	$=$
$R_{22}^{(0)}$	$(\epsilon + \mathbf{0} + \mathbf{1})$	$=$
$R_{11}^{(1)}$	$\epsilon + \mathbf{1} + (\epsilon + \mathbf{1})(\epsilon + \mathbf{1})^*(\epsilon + \mathbf{1})$	$= \mathbf{1^*}$
$R_{12}^{(1)}$	$\mathbf{0} + (\epsilon + \mathbf{1})(\epsilon + \mathbf{1})^*\mathbf{0}$	$= \mathbf{1^*0}$
$R_{21}^{(1)}$	$\emptyset + \emptyset(\epsilon + \mathbf{1})^*(\epsilon + \mathbf{1})$	$= \emptyset$
$R_{22}^{(1)}$	$\epsilon + \mathbf{0} + \mathbf{1} + \emptyset(\epsilon + \mathbf{1})^*\mathbf{0}$	$= \epsilon + \mathbf{0} + \mathbf{1}$
$R_{11}^{(2)}$	$\mathbf{1^*} + \mathbf{1^*0}(\epsilon + \mathbf{0} + \mathbf{1})^*\emptyset$	$= \mathbf{1^*}$
$R_{12}^{(2)}$	$\mathbf{1^*0} + \mathbf{1^*0}(\epsilon + \mathbf{0} + \mathbf{1})^*(\epsilon + \mathbf{0} + \mathbf{1})$	$= \mathbf{1^*0(0+1)^*}$
$R_{21}^{(2)}$	$\emptyset + (\epsilon + \mathbf{0} + \mathbf{1})(\epsilon + \mathbf{0} + \mathbf{1})^*\emptyset$	$= \emptyset$
$R_{22}^{(2)}$	$\epsilon + \mathbf{0} + \mathbf{1} + (\epsilon + \mathbf{0} + \mathbf{1})(\epsilon + \mathbf{0} + \mathbf{1})^*(\epsilon + \mathbf{0} + \mathbf{1}) = (\mathbf{0} + \mathbf{1})^*$	

State elimination algorithm

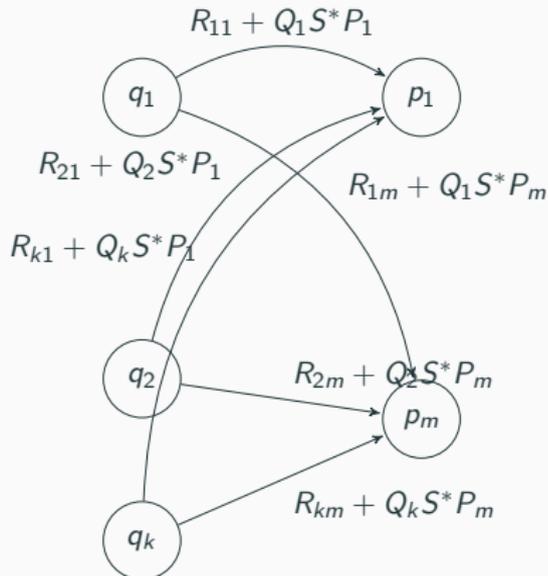
State elimination: the idea

Idea: Allow edges labelled by RE, iteratively remove nodes. (More efficient, avoids duplicity.)

State s selected for elimination



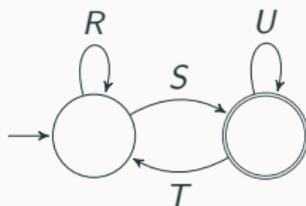
After s is eliminated.



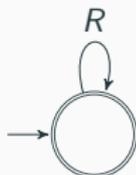
State elimination: the algorithm

For every accepting $q \in F$ eliminate all states $p \in Q \setminus \{q, q_0\}$.

- for $q \neq q_0$: $\text{RegE}(q) = (R + SU^*T)^*SU^*$



- for $q = q_0$: $\text{RegE}(q) = R^*$

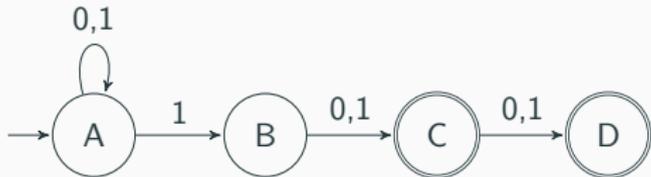


Finally, union over all accepting states: $\text{RegE}(A) = \sum_{q \in F} \text{RegE}(q)$

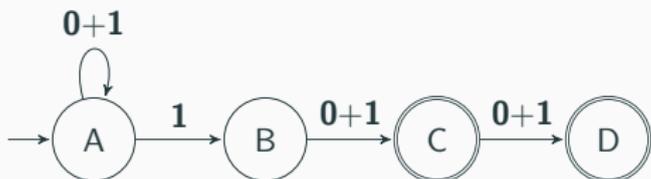
(Elimination order: first nonaccepting and noninitial states.)

State elimination: an example

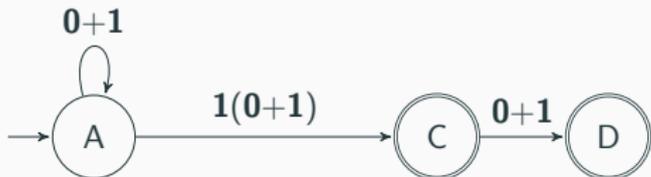
The original automaton:



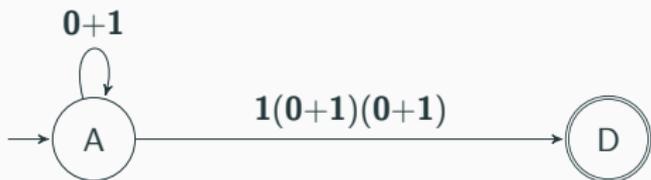
Replace letters by RE:



Eliminate B:



Eliminate C:



$$(0+1)^*1(0+1) + (0+1)^*1(0+1)(0+1)$$

Algebraic description of regular languages

Let $RL(\Sigma)$ denote the smallest set of languages over Σ that:

- contains \emptyset and $\{x\}$ for any letter $x \in \Sigma$, and
- is closed under union, concatenation, and iteration.

That is, for $A, B \in RL(\Sigma)$ also $A \cup B, A.B, A^* \in RL(\Sigma)$. Note that:

- $\{\epsilon\} \in RL(\Sigma)$ since $\{\epsilon\} = \emptyset^*$
- $\Sigma \in RL(\Sigma)$ since $\Sigma = \bigcup_{x \in \Sigma} \{x\}$ (a finite union)
- $\Sigma^* \in RL(\Sigma)$
- any finite language over Σ is in $RL(\Sigma)$.

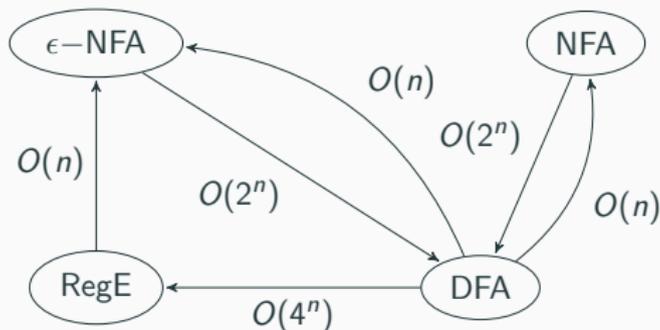
Theorem (A restatement of Kleene's Theorem)

A language over Σ is regular, iff it is in $RL(\Sigma)$.

Some properties to simplify RE (will not be tested)

$$\begin{aligned}L \cdot \emptyset = \emptyset \cdot L &= \emptyset \\ \{\epsilon\} \cdot L = L \cdot \{\epsilon\} &= L \\ (L^*)^* &= L^* \\ (L_1 \cup L_2)^* &= L_1^*(L_2 \cdot L_1^*)^* = L_2^*(L_1 \cdot L_2^*)^* \\ (L_1 \cdot L_2)^R &= L_2^R \cdot L_1^R \\ \partial_w(L_1 \cup L_2) &= \partial_w(L_1) \cup \partial_w(L_2) \\ \partial_w(\Sigma^* - L) &= \Sigma^* - \partial_w L.\end{aligned}$$

Converting between representations



- NFA or ϵ -NFA to DFA: $O(2^n)$
 - subset construction, DFA with up to 2^n states; for each state need $O(n^3)$ time to compute transitions.
 - ϵ -closure in $O(n^3)$ (search n states \times n^2 arcs)
- DFA to NFA or ϵ -NFA: $O(n)$
 - a simple modification of the transition table
- DFA to RE: $O(4^n)$
- RE to ϵ -NFA: $O(n)$

1.9 String substitution

String substitution and homomorphism

A (string) **substitution** is a mapping $\sigma: \Sigma^* \rightarrow \mathcal{P}(Y^*)$ where

- Σ and Y are finite alphabets, $Y = \bigcup_{x \in \Sigma} Y_x$
- for each $x \in \Sigma$, $\sigma(x)$ is a language over Y_x
- $\sigma(\epsilon) = \{\epsilon\}$ and $\sigma(u.v) = \sigma(u).\sigma(v)$

For a language $L \subseteq \Sigma^*$, $\sigma(L) = \bigcup_{w \in L} \sigma(w) \subseteq Y^*$. A substitution is **ϵ -free** if no $\sigma(x)$ contains ϵ .

A (string) **homomorphism** is defined similarly but each letter is mapped to a single word, $h: \Sigma^* \rightarrow Y^*$ where $h(x) \in Y_x^*$ for $x \in \Sigma$, $h(\epsilon) = \epsilon$ and $h(u.v) = h(u).h(v)$. Then $h(L) = \{h(w) \mid w \in L\}$. It is **ϵ -free** if $h(x) \neq \epsilon$ for all $x \in \Sigma$.

The **inverse homomorphism** applied to a language $L' \subseteq Y^*$:

$$h^{-1}(L') = \{w \in \Sigma^* \mid h(w) \in L'\}$$

Examples

Example (Substitution)

- If $\sigma(0) = \{a^i b^j, i, j \geq 0\}$ and $\sigma(1) = \{cd\}$, then $\sigma(010) = \{a^i b^j c d a^k b^l \mid i, j, k, l \geq 0\}$.
- $\Sigma = \{f, l, s, c, d\}$, $L = L((fsl)(cfsl)^*d)$ where
 - $\sigma(f)$ is a dictionary of first names
 - $\sigma(l)$ are last names
 - $\sigma(s) = \{ ' '\}$ (space), $\sigma(c) = \{ ', '\}$, $\sigma(d) = \{ '. '\}$
- A document template with symbols to be replaced by fields of database entries.

Example (Homomorphism)

- Define $h(0) = ab$ and $h(1) = \epsilon$. Then $h(0011) = abab$ and for $L = \mathbf{10^*1}$ we have $h(L) = L((ab)^*)$.
- Replace special symbols with T_EX code (e.g. $h(\mu) = \backslash mu$).

Theorem

Let $L \subseteq \Sigma^*$ be regular, $h: \Sigma^* \rightarrow Y^*$ a homomorphism, and $\sigma: \Sigma^* \rightarrow \mathcal{P}(Y^*)$ a substitution.

- The language $h(L)$ is regular.
- If $\sigma(x)$ is regular for all $x \in \Sigma$, then $\sigma(L)$ is also regular.

Moreover, if $L' \subseteq Y^*$ is regular, then $h^{-1}(L')$ is also regular.

Proof for homomorphism and substitution

Homomorphism \leftrightarrow substitution with $\sigma(x)$ one-element (regular).

Structural induction on a RE α such that $L = L(\alpha)$.

- **Induction base:** $\emptyset, \epsilon, \mathbf{a} \dots$ easy
- **Induction step:**

$$\sigma(L(\alpha + \beta)) = \sigma(L(\alpha)) \cup \sigma(L(\beta))$$

$$\sigma(L(\alpha\beta)) = \sigma(L(\alpha)).\sigma(L(\beta))$$

For iteration, decompose into an infinite union of powers:

$$\sigma(L(\alpha)^*) = \sigma(L(\alpha)^0) \cup \sigma(L(\alpha)^1) \cup \dots$$

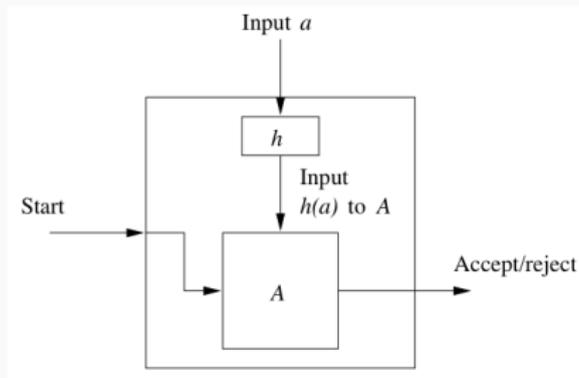
$$= \sigma(L(\alpha))^0 \cup \sigma(L(\alpha))^1 \cup \dots = \sigma(L(\alpha))^*$$

(Alternative view: take the tree of the RE α and replace every leaf x with a tree for a RE for $\sigma(x)$.)



Proof for inverse homomorphism & an example

Given a DFA $A = (Q, Y, \delta, q_0, F)$ recognizing L' , construct a DFA recognizing $h^{-1}(L')$: $B(Q, \Sigma, \delta_B, q_0, F)$ with $\delta_B(q, a) = \delta^*(q, h(a))$. That is, for a letter $a \in \Sigma$ do what A does for the word $h(a)$. Easy to show (by induction on $|w|$) that $\delta_B^*(q_0, w) = \delta^*(q_0, h(w))$.



□

Example (Inverse homomorphism)

$L' = L((00 + 1)^*)$, $h(a) = 01$, and $h(b) = 10$: $h^{-1}(L') = (ba)^*$.

$[h(L((ba)^*)) \in L']$ is obvious, other words generate an isolated 0.]

1.10 Decision properties of regular languages

Testing emptiness

Given a representation of a regular language L , is $L = \emptyset$?

FA: is any final state reachable from the initial state? $O(n^2)$

RE: convert to ϵ -NFA (in $O(n)$ time) and check reachability
or directly:

basis: \emptyset is empty, ϵ and \mathbf{a} are not

induction:

- $\alpha = (\alpha_1 + \alpha_2)$: empty iff both $L(\alpha_1)$ and $L(\alpha_2)$ are empty
- $\alpha = (\alpha_1\alpha_2)$ empty iff either $L(\alpha_1)$ or $L(\alpha_2)$ is empty
- $\alpha = (\alpha_1^*)$ never empty, includes ϵ

Testing membership

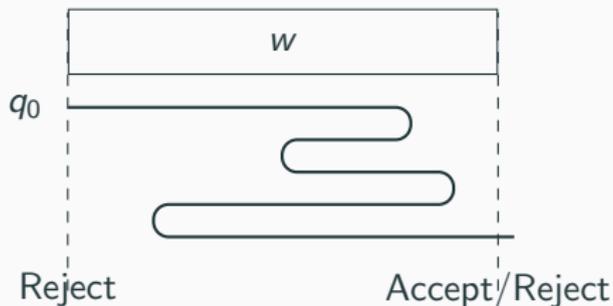
Given a regular language L and a word w , is $w \in L$?

- **DFA**: run the automaton; if $|w| = n$, with a suitable representation (constant time transitions) it is in $O(n)$
- **NFA** with s states: running time $O(ns^2)$, each letter processed by taking the previous set of states
- **ϵ -NFA**: first compute the ϵ -closure, then for each letter, process it and compute the ϵ -closure of the result
- **RE** of size s : convert to an ϵ -NFA with at most $2s$ states and then simulate, $O(ns^2)$

1.11 Two-way finite automata

Revisit previous input?

- Finite automaton makes following actions:
 - read a symbol
 - changes its state
 - moves its reading head to the right
- The head is not allowed to move to the left.
- What happens, if we allow the head to move left and right?
- **The automaton does not write anything on the tape!**



Two-way finite automaton

Definition

A **Two-way DFA** is a five-tuple $A = (Q, \Sigma, \delta, q_0, F)$, where Q, Σ, q_0, F are as in the definition of a DFA, and the transition function is a mapping

$$\delta : Q \times \Sigma \rightarrow Q \times \{-1, +1\}.$$

Similarly, a **Two-way NFA** is defined as an NFA with

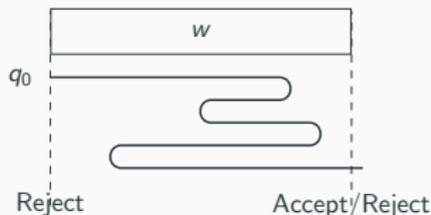
$$\delta : Q \times \Sigma \rightarrow \mathcal{P}(Q \times \{-1, +1\}).$$

- the extra value indicates direction of head movement: -1 for left, $+1$ for right
- we could allow for no movement, i.e., $\{-1, 0, +1\}$

Two-way DFA computation

A word w is **accepted** by a two-way DFA, if:

- computation started in q_0 at the first letter of w
- the head never moved left from the first letter of w
- the head moved right from the last letter of w
- we ended in an accepting state



NB: Infinite cycles are possible.

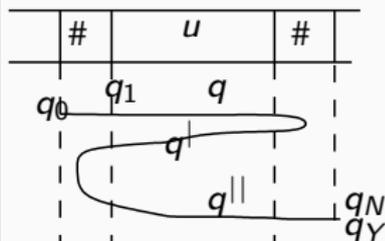
Idea: We can bookend all words with a special symbol $\# \notin \Sigma$. If $L(A) = \{\#w\# \mid w \in L\}$ is regular, so is L because $L = \partial_{\#} \partial_{\#}^R L(A)$

Example

Let $A = (Q, \Sigma, \delta, q_1, F)$. Define a 2-way DFA B recognizing $L(B) = \{\#u\# \mid uu \in L(A)\}$ (this is not left/right quotient!)

$$B = (Q \cup Q^l \cup Q^r \cup \{q_0, q_N, q_Y\}, \Sigma, \delta^l, q_0, \{q_Y\})$$

δ^l	$x \in \Sigma$	#	remark
q_0	$q_N, -1$	$q_1, +1$	q_1 is starting in A
q	$p, +1$	$q^l, -1$	$p = \delta(q, x)$
q^l	$q^l, -1$	$q^r, +1$	
q^r	$p^r, +1$	$q_Y, +1$	$q \in F, p = \delta(q, x)$
q^r	$p^r, +1$	$q_N, +1$	$q \notin F, p = \delta(q, x)$
q_N	$q_N, +1$	$q_N, +1$	
q_Y	$q_N, +1$	$q_N, +1$	



Two-way finite automata and regular Languages

Theorem

Two-way DFA recognize exactly regular languages.

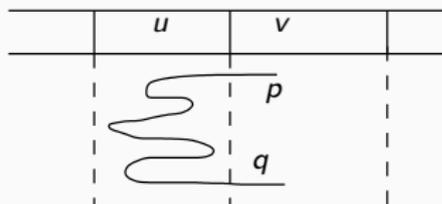
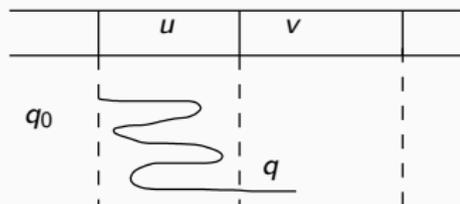
⇐ Convert DFA trivially to two-way DFA: $\delta'(q, a) = (\delta(q, a), +1)$

⇒ Use Myhill-Nerode theorem: associate with each word $u \in \Sigma^*$ a function $f_u : Q \cup \{q_0\} \rightarrow Q \cup \{\perp\}$ describing the influence of a subword u on the computation.

How exactly? See the next slide.

There are only finitely many, at most $(|Q| + 1)^{|Q|+1}$ such functions. Define $u \sim v$ iff $f_u = f_v$, verify that it is a right congruence and $L(A)$ is a union of classes. □

How $f_u : Q \cup \{q_0^\perp\} \rightarrow Q \cup \{\perp\}$ describes computation over u



$f_u(q_0^\perp) =$ state after first transition right in case we begin left in state q_0

$f_u(p) =$ state of right transition in case we begin right in state p

\perp denotes failure (a cycle or escape left)

Summary of finite automata

- Finite Automata: DFA, reduced DFA, NFA, ϵ -NFA
- Regular Expressions
- Regular languages: closed under set operations, string operations, substitution, homomorphism, inverse hom.
- all FA and RE describe the same class of languages
- Key theorems:
 - Myhill–Nerode (implicit DFA via congruences on words)
 - Kleene (regular languages iff matched by RE)
 - Pumping lemma
- Two-way automata
- (optional) Automata with output

Summary of Lecture 4

- regular expressions
- Kleene's theorem (two variants)
- constructions: RE to ϵ -NFA, DFA to RE
- state elimination algorithm
- string substitution, homomorphism, inverse homomorphism
- decision properties
- two-way finite automata