NTIN071 A&G: Tutorial 2 – Pumping lemma, Myhill–Nerode theorem, Equivalent and minimal representations

Teaching goals: The student is able to

- state and prove the Pumping lemma for regular languages
- apply the Pumping lemma to prove nonregularity of a given language
- state and prove the Myhill-Nerode theorem
- apply the Myhill-Nerode theorem to prove regularity, to construct a DFA
- apply the Myhill-Nerode theorem to prove nonregularity
- define reachability & equivalence of states, reduced DFA, automata homomorphism
- apply the reachable states and state equivalence algorithms to reduce a DFA

In-class problems

Problem 1 (Pumping Lemma: statement)**.** (a) Formulate the Pumping Lemma for regular languages (without consulting your notes).

(b) How is the number n form its statement related to a recognizing automaton?

(c) Prove it (without consulting your notes).

(d) Demonstrate pumping on $L = \{w \in \{a,b\}^* \mid w \text{ contains } abba \text{ as a subword}\}$.

Problem 2 (Pumping Lemma: application)**.** Use the Pumping Lemma to prove that the following languages are not regular. (The alphabet is $\Sigma = \{a, b\}$.)

(a) $L = \{a^i b^j \mid i \geq j\}$

(b) $L = \{a^{i^2} \mid i \geq 0\}$

(c) $L = \{a^i b^{i+j} a^j \mid i, j \geq 0\}$

(d) $L = \{ww^R \mid w \in \Sigma^*\}$, where w^R is w reversed

Problem 3 (Myhill–Nerode theorem: statement)**.** (a) Formulate the Myhill–Nerode theorem and recall the idea of its proof (without consulting your notes).

(b) Show that if we forget any of the conditions on the equivalence $\sim$, the resulting statement is not true.

Problem 4 (Myhill–Nerode theorem: application)**.** Prove or disprove using the Myhill–Nerode theorem that the following languages are regular.

(a) $L = \{aa, ab, ba\}$

(b) $L = \{a^i b^j \mid i \geq j\}$

(c) $L = \{a^{i^2} \mid i \geq 0\}$

(d) $L = \{ww^R \mid w \in \Sigma^*\}$, where w^R is w reversed

(e) $L = \{a^i b^{i+j} a^j \mid i, j \geq 0\}$

Problem 5 (Equivalent and minimal representations)**.** For the automata below:
(a) Find and remove all unreachable states.
(b) Determine the state equivalence (indistinguishability) relations. (Moreover, for any distinguishable pair of states find all minimal-length distinguishing words.)
(c) Construct their reducts.

A	a	b
$\to * 0$	1	2
1	3	0
2	4	5
3	0	2
4	2	5
5	0	3

B	a	b
$\to * 0$	0	5
1	1	3
2	2	7
3	3	2
$* 4$	6	1
5	5	1
$* 6$	4	2
7	7	0

C	a	b
$\to 0$	1	2
1	1	3
$* 2$	2	4
3	1	6
$* 4$	5	2
$* 5$	5	5
6	6	3
7	1	2
8	8	3

D	a	b
$* 0$	7	6
1	1	0
2	4	3
3	3	1
4	2	3
5	5	4
6	6	5
$\to * 7$	0	6

Extra Practice and Thinking

Problem 6. Use the Pumping Lemma to prove that the following languages are not regular. (The alphabet is $\Sigma = \{a, b\}$.)
(a) $L = \{a^i b^j \mid i \le j\}$
(b) $L = \{a^{2^i} \mid i \ge 0\}$
(c) $L = \{ww \mid w \in \Sigma^*\}$

Problem 7. Prove or disprove using the Myhill–Nerode theorem that the following languages are regular.
(a) $L = \{a^i b^j \mid i \le j\}$
(b) $L_k = \{a^i b^j \mid i \le j \le k\}$ for a fixed $k \in \mathbb{N}$
(c) $L = \{a^{2^i} \mid i \ge 0\}$
(d) $L = \{ww \mid w \in \Sigma^*\}$

Problem 8 (Pumping Lemma: generalization)**.** (a) Can we change the condition $|uv| \le n$ with $|vw| \le n$, that is, *iterate near the end*? Prove or disprove.
(b) Can we iterate near a chosen position in the word? How to formulate (and prove) such a generalization?

Problem 9 (Equivalences on words)**.** Give an example of an equivalence relation on Σ^* which:
(a) is a right and a left congruence
(b) is a right but not a left congruence
(c) is of finite index